A Topos Foundation for Theories of Physics: IV. Categories of Systems

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Abstract

This paper is the fourth in a series whose goal is to develop a fundamentally new way of building theories of physics. The motivation comes from a desire to address certain deep issues that arise in the quantum theory of gravity. Our basic contention is that constructing a theory of physics is equivalent to finding a representation in a topos of a certain formal language that is attached to the system. Classical physics arises when the topos is the category of sets. Other types of theory employ a different topos.

The previous papers in this series are concerned with implementing this programme for a single system. In the present paper, we turn to considering a *collection* of systems: in particular, we are interested in the relation between the topos representation for a composite system, and the representations for its constituents. We also study this problem for the disjoint sum of two systems. Our approach to these matters is to construct a *category* of systems and to find a topos representation of the entire category.

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1 Introduction

This is the fourth in a series of papers whose aim is to construct a general framework within which theories of physics can be expressed in a topos other than that of sets. In the first paper, [1], we developed the idea of a typed, higher-order (local) language, $\mathcal{L}_S$, for each physical system S, with the goal of finding representations of this language in various topoi. Then, in the second and third papers, [2, 3], we showed in detail how the 'daseinisation' operation in quantum theory enables us to represent this language—and a simpler propositional language—in a certain topos of presheaves.

In the present paper, we return to the more general aspects of our theory, and study its application to a *collection* of systems, each one of which may be associated with a different topos. For example, if S_1, S_2 is a pair of systems, with associated topoi $\tau(S_1)$ and $\tau(S_2)$, and if S_1 is a sub-system of S_2, then we wish to consider how $\tau(S_1)$ is related to $\tau(S_2)$. Similarly, if a composite system is formed from a pair of systems S_1, S_2, what relations are there between the topos of the composite system and the topoi of the constituent parts?

We start in Section 2, by introducing the notion of a 'category of systems', **Sys**, whose objects are the physical systems of interest, and whose arrows represent situations in which one system is a 'sub-system', or a 'constituent' of another, or combinations of such situations. A particular example of a sub-system arises in the 'disjoint sum' of two systems. We argue on physical grounds that **Sys** can be regarded as a symmetric monoidal category in two ways: one in which the monoidal product represents forming a composite system, and one in which the monoidal product represents the disjoint sum. We show how such arrows correspond to 'translations' of the local languages associated with the component systems. We also give a preliminary definition of the representation of the category **Sys** in a category of topoi, $\mathcal{M}(\mathbf{Sys})$. We then show that the scheme works consistently in classical physics.

The idea of representing **Sys** is developed at length in Section 3. An important ingredient is the 'pull-back' operation that arises when representing the arrows of **Sys** in the category of topoi, $\mathcal{M}(\mathbf{Sys})$. Then, in Section 3.2 we bring together all these ideas in the form of a set of rules for constructing a topos representation of the objects and arrows in the Category **Sys**.

In Section 4 we show how our earlier work (in papers II and III) on toposifying quantum theory can be extended to give a topos representation of **Sys**. The disjoint sum of systems behaves well under the pull-back operation but the situation for the composition of systems is different: something, we think, that reflects the existence of entanglement in quantum theory. Finally, in Section 5 we speculate a little on how the general scheme might be developed.

2 The Category of Systems

2.1 Background Remarks

In one sense, there is only one true 'system', and that is the universe as a whole. Concomitantly, there is just one local language, and one topos. However, in practice, the universe is divided conceptually into portions that are sufficiently simple to be amenable to theoretical discussion. Of course, this division is not unique, but it must be such that the coupling between portions is weak enough that, to a good approximation, their theoretical models can be studied in isolation from each other. Such an essentially isolated[3] portion of the universe is called a 'sub-system'. By an abuse of language, sub-systems of the universe are usually called 'systems' (so that the universe as a whole is one super-system), and then we can talk about 'sub-systems' of these systems; or 'composites' of them; or sub-systems of the composite systems, and so on.

Of course, in practice, references by physicists to systems and sub-systems[4] do not generally signify *actual* sub-systems of the real universe but rather idealisations of possible systems. This is what a physics lecturer means when he or she starts a lecture by saying "Consider a point particle moving in three dimensions.....".

To develop these ideas further we need mathematical control over the systems of interest, and their interrelations. To this end, we start by focussing on some collection, **Sys**, of physical systems to which a particular theory-type is deemed to be applicable. For example, we could consider a collection of systems that are to be discussed using the methodology of classical physics; or systems to be discussed using standard quantum theory; or whatever. For completeness, we require that every sub-system of a system in **Sys** is itself a member of **Sys**, as is every composite of members of **Sys**.

We shall assume that the systems in **Sys** are all associated with local languages of the type discussed in paper I, and that they all have the *same* set of ground symbols which, for the purposes of the present discussion, we take to be just Σ and $\mathcal{R}$. It follows that the languages $\mathcal{L}(S)$, $S \in \mathbf{Sys}$, differ from each other only in the set of function symbols $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$; i.e., the set of *physical quantities*.

As a simple example of the system-dependence of the set of function symbols let system S_1 be a point particle moving in one dimension, and let the set of physical quantities be $F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) = \{x, p, H\}$. In the language $\mathcal{L}(S_1)$, these function-symbols represent the position, momentum, and energy of the system respectively. On the other hand, if S_2 is a particle moving in three dimensions, then in the language $\mathcal{L}(S_2)$ we could have $F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big) = \{x, y, z, p_x, p_y, p_z, H\}$ to allow for three-dimensional position and momentum. Or, we could decide to add angular momentum as well, to give the

[3]The ideal monad has no windows.

[4]The word 'sub-system' does not only mean a collection of objects that is spatially localised. One could also consider sub-systems of field systems by focussing on a just a few modes of the fields as is done, for example, in the Robertson-Walker model for cosmology. Another possibility would be to use fields localised in some fixed space, or space-time region provided that this is consistent with the dynamics.

set $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, J_x, J_y, J_z, H\}$.

2.2 The Category Sys

2.2.1 The Arrows and Translations for the Disjoint Sum $S_1 \sqcup S_2$.

The use of local languages is central to our overall topos scheme, and therefore we need to understand, in particular, (i) the relation between the languages $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ if S_1 is a sub-system of S_2; and (ii) the relation between $\mathcal{L}(S_1)$, $\mathcal{L}(S_2)$ and $\mathcal{L}(S_1 \diamond S_2)$, where $S_1 \diamond S_2$ denotes the composite of systems S_1 and S_2.

These discussions can be made more precise by regarding **Sys** as a category whose objects are the systems.[5] The arrows in **Sys** need to cover two basic types of relation: (i) that between S_1 and S_2 if S_1 is a 'sub-system' of S_2; and (ii) that between a composite system, $S_1 \diamond S_2$, and its constituent systems, S_1 and S_2.

This may seem straightforward but, in fact, care is needed since although the idea of a 'sub-system' seems intuitively clear, it is hard to give a physically acceptable definition that is universal. However, some insight into this idea can be gained by considering its meaning in classical physics. This is very relevant for the general scheme since one of our main goals is to make all theories 'look' like classical physics in the appropriate topos.

To this end, let S_1 and S_2 be classical systems whose state spaces are the symplectic manifolds $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. If S_1 is deemed to be a sub-system of S_2, it is natural to require that $\mathcal{S}_1$ is a *sub-manifold* of $\mathcal{S}_2$, i.e., $\mathcal{S}_1 \subseteq \mathcal{S}_2$. However, this condition cannot be used as a *definition* of a 'sub-system' since the converse may not be true: i.e., if $\mathcal{S}_1 \subseteq \mathcal{S}_2$, this does not necessarily mean that, from a physical perspective, S_1 could, or would, be said to be a sub-system of S_2.[6]

On the other hand, there are situations where being a sub-manifold clearly *does* imply being a physical sub-system. For example, suppose the state space $\mathcal{S}$ of a system S is a disconnected manifold with two components $\mathcal{S}_1$ and $\mathcal{S}_2$, so that $\mathcal{S}$ is the disjoint union, $\mathcal{S}_1 \coprod \mathcal{S}_2$, of the sub-manifolds $\mathcal{S}_1$ and $\mathcal{S}_2$. Then it seems physically appropriate to say that the system S itself is disconnected, and to write $S = S_1 \sqcup S_2$ where the symplectic manifolds that represent the sub-systems S_1 and S_2 are $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively.

One reason why it is reasonable to call S_1 and S_2 'sub-systems' in this particular situation is that any continuous dynamical evolution of a state point in $\mathcal{S} \simeq \mathcal{S}_1 \sqcup \mathcal{S}_2$ will always lie in either one component or the other. This suggests that perhaps, in general,

[5]To control the size of **Sys** we assume that the collection of objects/systems is a *set* rather than a more general class.

[6] For example, consider the diagonal sub-manifold $\Delta(\mathcal{S}) \subset \mathcal{S} \times \mathcal{S}$ of the symplectic manifold $\mathcal{S} \times \mathcal{S}$ that represents the composite $S \diamond S$ of two copies of a system S. Evidently, the states in $\Delta(\mathcal{S})$ correspond to the situation in which both copies of S 'march together'. It is doubtful if this would be recognised physically as a sub-system.

a necessary condition for a sub-manifold $\mathcal{S}_1 \subseteq \mathcal{S}_2$ to represent a physical sub-system is that the dynamics of the system S_2 must be such that $\mathcal{S}_1$ is mapped into itself under the dynamical evolution on $\mathcal{S}_2$; in other words, $\mathcal{S}_1$ is a *dynamically-invariant* sub-manifold of $\mathcal{S}_2$. This correlates with the idea mentioned earlier that sub-systems are weakly-coupled with each other.

However, such a dynamical restriction is not something that should be coded into the languages, $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$: rather, the dynamics is to be associated with the *representation* of these languages in the appropriate topoi.

Still, this caveat does not apply to the disjoint sum $S_1 \sqcup S_2$ of two systems S_1, S_2, and we will assume that, in general, (i.e., not just in classical physics) it is legitimate to think of S_1 and S_2 as being sub-systems of $S_1 \sqcup S_2$; something that we indicate by defining arrows $i_1 : S_1 \to S_1 \sqcup S_2$, and $i_2 : S_2 \to S_1 \sqcup S_2$ in **Sys**.

To proceed further it is important to understand the connection between the putative arrows in the category **Sys**, and the 'translations' of the associated languages. The first step is to consider what can be said about the relation between $\mathcal{L}(S_1 \sqcup S_2)$, and $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$. All three languages share the same ground-type symbols, and so what we are concerned with is the relation between the function symbols of signature $\Sigma \to \mathcal{R}$ in these languages.

By considering what is meant intuitively by the disjoint sum, it seems plausible that each physical quantity for the system $S_1 \sqcup S_2$ produces a physical quantity for S_1, and another one for S_2. Conversely, specifying a pair of physical quantities—one for S_1 and one for S_2—gives a physical quantity for $S_1 \sqcup S_2$. In other words,

$$F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \simeq F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \times F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big) \tag{2.1}$$

However, it is important not to be too dogmatic about statements of this type since in non-classical theories new possibilities can arise that are counter to intuition.

Associated with (2.1) are the maps $\mathcal{L}(i_1) : F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ and $\mathcal{L}(i_2) : F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big)$, defined as the projection maps of the product. In the theory of local languages, these transformations are essentially *translations* [5] of $\mathcal{L}(S_1 \sqcup S_2)$ in $\mathcal{L}(S_1)$ and $\mathcal{L}(S_2)$ respectively; a situation that we denote $\mathcal{L}(i_1) : \mathcal{L}(S_1 \sqcup S_2) \to \mathcal{L}(S_1)$, and $\mathcal{L}(i_2) : \mathcal{L}(S_1 \sqcup S_2) \to \mathcal{L}(S_2)$.

To be more precise, these operations are translations if, taking $\mathcal{L}(i_1)$ as the explanatory example, the map $\mathcal{L}(i_1) : F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ is supplemented with the following map from the ground symbols of $\mathcal{L}(S_1 \sqcup S_2)$ to those of $\mathcal{L}(S_1)$:

$$\mathcal{L}(i_1)(\Sigma) := \Sigma, \tag{2.2}$$
$$\mathcal{L}(i_1)(\mathcal{R}) := \mathcal{R}, \tag{2.3}$$
$$\mathcal{L}(i_1)(1) := 1, \tag{2.4}$$
$$\mathcal{L}(i_1)(\Omega) := \Omega. \tag{2.5}$$

Such a translation map is then extended to all type symbols using the definitions

$$\mathcal{L}(i_1)(T_1 \times T_2 \times \cdots \times T_n) = \mathcal{L}(i_1)(T_1) \times \mathcal{L}(i_1)(T_2) \times \cdots \times \mathcal{L}(i_1)(T_n), \tag{2.6}$$
$$\mathcal{L}(i_1)(PT) = P[\mathcal{L}(i_1)(T)] \tag{2.7}$$

for all finite n and all type symbols $T, T_1, T_2, \ldots, T_n$. This, in turn, can be extended inductively to all terms in the language. Thus, in our case, the translations act trivially on all the type symbols.

Arrows in Sys *are* translations. Motivated by this argument we now turn everything around and, in general, *define* an arrow $j : S_1 \to S$ in the category **Sys** to mean that there is some *physically meaningful* way of transforming the physical quantities in S to physical quantities in S_1. If, for any pair of systems S_1, S there is more than one such transformation, then there will be more than one arrow from S_1 to S.

To make this more precise, let **Loc** denote the collection of all (small[7]) local languages. This is a category whose objects are the local languages, and whose arrows are translations between languages. Then our basic assumption is that the association $S \mapsto \mathcal{L}(S)$ is a covariant functor from **Sys** to $\mathbf{Loc}^{\mathrm{op}}$, which we denote as $\mathcal{L} : \mathbf{Sys} \to \mathbf{Loc}^{\mathrm{op}}$.

Note that the combination of a pair of arrows in **Sys** exists in so far as the associated translations can be combined.

2.2.2 The Arrows and Translations for the Composite System $S_1 \diamond S_2$.

Let us now consider the composition $S_1 \diamond S_2$ of a pair of systems. In the case of classical physics, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are the symplectic manifolds that represent the systems S_1 and S_2 respectively, then the manifold that represents the composite system is the cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$. This is distinguished by the existence of the two projection functions $\mathrm{pr}_1 : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_1$ and $\mathrm{pr}_2 : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_2$.

It seems reasonable to impose the same type of structure on **Sys**: i.e., to require there to be arrows $p_1 : S_1 \diamond S_2 \to S_1$ and $p_2 : S_1 \diamond S_2 \to S_2$ in **Sys**. However, bearing in mind the definition above, these arrows p_1, p_2 exist if, and only if, there are corresponding translations $\mathcal{L}(p_1) : \mathcal{L}(S_1) \to \mathcal{L}(S_1 \diamond S_2)$, and $\mathcal{L}(p_2) : \mathcal{L}(S_2) \to \mathcal{L}(S_1 \diamond S_2)$. But there *are* such translations: for if A_1 is a physical quantity for system S_1, then $\mathcal{L}(p_1)(A_1)$ can be defined as that same physical quantity, but now regarded as pertaining to the combined system $S_1 \diamond S_2$; and analogously for system S_2.[8] We shall denote this translated quantity, $\mathcal{L}(p_1)(A_1)$, by $A_1 \diamond 1$.

[7]This means that the collection of symbols is a set, not a more general class.

[8]For example, if A is the energy of particle 1, then we can talk about this energy in the combination of a pair of particles. Of course, in—for example—classical physics there is no reason why the energy of particle 1 should be *conserved* in the composite system, but that, dynamical, question is a different matter.

Note that we do *not* postulate any simple relation between $F_{\mathcal{L}(S_1 \diamond S_2)}(\Sigma, \mathcal{R})$ and $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ and $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$; i.e., there is no analogue of (2.1) for combinations of systems.

The definitions above of the basic arrows suggest that we might also want to impose the following conditions:

1. The arrows $i_1 : S_1 \to S_1 \sqcup S_2$, and $i_2 : S_2 \to S_1 \sqcup S_2$ are *monic* in **Sys**.

2. The arrows $p_1 : S_1 \diamond S_2 \to S_1$ and $p_2 : S_1 \diamond S_2 \to S_2$ are *epic* arrows in **Sys**.

However, we do *not* require that $S_1 \cup S_2$ and $S_1 \diamond S_2$ are the co-product and product, respectively, of S_1 and S_2 in the category **Sys**.

2.2.3 The Concept of 'Isomorphic' Systems.

We also need to decide what it means to say that two systems S_1 and S_2 are *isomorphic*, to be denoted $S_1 \simeq S_2$. As with the concept of sub-system, the notion of isomorphism is to some extent a matter of definition rather than obvious physical structure, albeit with the expectation that isomorphic systems in **Sys** will correspond to isomorphic local languages, and be represented by isomorphic mathematical objects in any concrete realisation of the axioms: for example, by isomorphic symplectic manifolds in classical physics.

To a considerable extent, the physical meaning of 'isomorphism' depends on whether one is dealing with actual physical systems, or idealisations of them. For example, an electron confined in a box in Cambridge is presumably isomorphic to one confined in the same type of box in London, although they are not the same physical system. On the other hand, when a lecturer says "Consider an electron trapped in a box....", he/she is referring to an idealised system.

One could, perhaps, say that an idealised system is an *equivalence class* (under isomorphisms) of real systems, but even working only with idealisations does not entirely remove the need for the concept of isomorphism.

For example, in classical mechanics, consider the (idealised) system S of a point particle moving in a box, and let 1 denote the 'trivial system' that consists of just a single point with no internal or external degrees of freedom. Now consider the system $S \diamond 1$. In classical mechanics this is represented by the symplectic manifold $\mathcal{S} \times \{*\}$, where $\{*\}$ is a single point, regarded as a zero-dimensional manifold. However, $\mathcal{S} \times \{*\}$ is isomorphic to the manifold $\mathcal{S}$, and it is clear physically that the system $S \diamond 1$ is isomorphic to the system S. On the other hand, one cannot say that $S \diamond 1$ is literally *equal* to S, so the concept of 'isomorphism' needs to be maintained.

One thing that *is* clear is that if $S_1 \simeq S_2$ then $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \simeq F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$, and if any other non-empty sets of function symbols are present, then they too must be isomorphic.

Note that when introducing a trivial system, 1, it necessary to specify its local language, $\mathcal{L}(1)$. The set of function symbols $F_{\mathcal{L}(1)}\big(\Sigma, \mathcal{R}\big)$ is not completely empty since, in classical physics, one does have a preferred physical quantity, which is just the number 1. If one asks what is meant in general by the 'number 1' the answer is not trivial since, in the reals $\mathbb{R}$, the number 1 is the multiplicative identity. It would be possible to add the existence of such a unit to the axioms for $\mathcal{R}$ but this would involve introducing a multiplicative structure and we do not know if there might be physically interesting topos representations that do not have this feature.

For the moment then, we will say that the trivial system has just a single physical quantity, which in classical physics translates to the number 1. More generally, for the language $\mathcal{L}(1)$ we specify that $F_{\mathcal{L}(1)}\big(\Sigma, \mathcal{R}\big) := \{I\}$, i.e., $F_{\mathcal{L}(1)}\big(\Sigma, \mathcal{R}\big)$ has just a single element, I, say. Furthermore, we add the axiom

$$: \forall \tilde{s}_1 \forall \tilde{s}_2, I(\tilde{s}_1) = I(\tilde{s}_2), \tag{2.8}$$

where $\tilde{s}_1$ and $\tilde{s}_2$ are variables of type Σ. In fact, it seems natural to add such a trivial quantity to the language $\mathcal{L}(S)$ for *any* system S, and from now on we will assume that this has been done.

A related issue is that, in classical physics, if A is a physical quantity, then so is rA for any $r \in \mathbb{R}$. This is because the set of classical quantities $A_\sigma : \Sigma_\sigma \to \mathcal{R}_\sigma \simeq \mathbb{R}$ forms a *ring* whose structure derives from the ring structure of $\mathbb{R}$. It would be possible to add ring axioms for $\mathcal{R}$ to the language $\mathcal{L}(S)$, but we think this is too strong, not least because, as shown in paper III, it fails in quantum theory [3]. Clearly, the general question of axioms for $\mathcal{R}$ needs more thought: a task for later work.

If desired, an '*empty*' system, 0, can be added too, with $F_{\mathcal{L}(0)}\big(\Sigma, \mathcal{R}\big) := \emptyset$. This, so called, 'pure language', $\mathcal{L}(0)$, is an initial object in the category **Loc**.

2.2.4 An Axiomatic Formulation of Sys

Let us now summarise, and clarify, our list of axioms for a category **Sys**:

1. The collection **Sys** is a small category where (i) the objects are the systems of interest (or, if desired, isomorphism classes of such systems); and (ii) the arrows are defined as above.

   Thus the fundamental property of an arrow $j : S_1 \to S$ in **Sys** is that it induces, and is essentially *defined by*, a translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$. Physically, it corresponds to the physical quantities for system S being 'pulled-back' to give physical quantities for system S_1.

   Arrows of particular interest are those associated with 'sub-systems' and 'composite systems', as discussed above.

2. The axioms for a category are satisfied because:

(a) Physically, the ability to form composites of arrows follows from the concept of 'pulling-back' physical quantities. From a mathematical perspective, if $j : S_1 \to S_2$ and $k : S_2 \to S_3$, then the translations give functions $\mathcal{L}(j) : F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ and $\mathcal{L}(k) : F_{\mathcal{L}(S_3)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big)$. Then clearly $\mathcal{L}(j) \circ \mathcal{L}(k) : F_{\mathcal{L}(S_3)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$, and this can thought of as the translation corresponding to the arrow $k \circ j : S_1 \to S_3$.

The associativity of the law of arrow combination can be proved in a similar way.

(b) We add by hand a special arrow $\mathrm{id}_S : S \to S$ which is defined to correspond to the translation $\mathcal{L}(\mathrm{id}_S)$ that is given by the identity map on $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$. Clearly, $\mathrm{id}_S : S \to S$ acts an an identity morphism should.

3. For any pair of systems S_1, S_2, there is a *disjoint sum*, denoted $S_1 \sqcup S_2$. The disjoint sum has the following properties:

(a) For all systems S_1, S_2, S_3 in **Sys**:

$$(S_1 \sqcup S_2) \sqcup S_3 \simeq S_1 \sqcup (S_2 \sqcup S_3). \tag{2.9}$$

(b) For all systems S_1, S_2 in **Sys**:

$$S_1 \sqcup S_2 \simeq S_2 \sqcup S_1. \tag{2.10}$$

(c) There are arrows in **Sys**:

$$i_1 : S_1 \to S_1 \sqcup S_2 \ \text{ and } i_2 : S_2 \to S_1 \sqcup S_2 \tag{2.11}$$

that are associated with translations in the sense discussed in Section 2.2.1. These are associated with the decomposition

$$F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \simeq F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \times F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big). \tag{2.12}$$

We assume that if S_1, S_2 belong to **Sys**, then **Sys** also contains $S_1 \sqcup S_2$.

4. For any given pair of systems S_1, S_2, there is a *composite* system in **Sys**, denoted[9] $S_1 \diamond S_2$, with the following properties:

(a) For all systems S_1, S_2, S_3 in **Sys**:

$$(S_1 \diamond S_2) \diamond S_3 \simeq S_1 \diamond (S_2 \diamond S_3). \tag{2.13}$$

(b) For all systems S_1, S_2 in **Sys**:

$$S_1 \diamond S_2 \simeq S_2 \diamond S_1. \tag{2.14}$$

[9]The product operation in a monoidal category is often written '$\otimes$'. However, a different symbol has been used here to avoid confusion with existing usages in physics of the tensor product sign '$\otimes$'.

(c) There are arrows in **Sys**:

$$p_1 : S_1 \diamond S_2 \to S_1 \text{ and } p_2 : S_1 \diamond S_2 \to S_2 \tag{2.15}$$

that are associated with translations in the sense discussed in Section 2.2.2.

We assume that if S_1, S_2 belong to **Sys**, then **Sys** also contains the composite system $S_1 \diamond S_2$.

5. It seems physically reasonable to add the axiom

$$(S_1 \sqcup S_2) \diamond S \simeq (S_1 \diamond S) \sqcup (S_2 \diamond S) \tag{2.16}$$

for all systems S_1, S_2, S. However, physical intuition can be a dangerous thing, and so, as with most of these axioms, we are not dogmatic, and feel free to change them as new insights emerge.

6. There is a trivial system, 1, such that for all systems S, we have

$$S \diamond 1 \simeq S \simeq 1 \diamond S \tag{2.17}$$

7. It may be convenient to postulate an 'empty system', 0, with the properties

$$S \diamond 0 \simeq 0 \diamond S \simeq 0 \tag{2.18}$$
$$S \sqcup 0 \simeq 0 \sqcup S \simeq S \tag{2.19}$$

for all systems S.

Within the meaning given to arrows in **Sys**, 0 is a *terminal object* in **Sys**. This is because the empty set of function symbols of signature $\Sigma \to \mathcal{R}$ is a subset of any other set of function symbols of this signature.

It might seem tempting to postulate that composition laws are well-behaved with respect to arrows. Namely, if $j : S_1 \to S_2$, then, for any S, there is an arrow $S_1 \diamond S \to S_2 \diamond S$ and an arrow $S_1 \sqcup S \to S_2 \sqcup S$.[10]

In the case of the disjoint sum, such an arrow can be easily constructed using (2.12). First split the function symbols in $F_{\mathcal{L}(S_1 \sqcup S)}(\Sigma, \mathcal{R})$ into $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ and the function symbols in $F_{\mathcal{L}(S_2 \sqcup S)}(\Sigma, \mathcal{R})$ into $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$. Since there is an arrow $j : S_1 \to S_2$, there is a translation $\mathcal{L}(j) : \mathcal{L}(S_2) \to \mathcal{L}(S_1)$, given by a mapping $\mathcal{L}(j) : F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \to F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$. Of course, then there is also a mapping $\mathcal{L}(j) \times \mathcal{L}(\mathrm{id}_S) : F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \to F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$, i.e., a translation between $\mathcal{L}(S_2 \sqcup S)$ and $\mathcal{L}(S_1 \sqcup S)$. Since we assume that there is an arrow

[10] A more accurate way of capturing this idea is to say that the operation **Sys** $\times$ **Sys** $\to$ **Sys** in which

$$\langle S_1, S_2 \rangle \mapsto S_1 \diamond S_2 \tag{2.20}$$

is a *bi-functor* from **Sys** $\times$ **Sys** to **Sys**. Ditto for the operation in which $\langle S_1, S_2 \rangle \mapsto S_1 \sqcup S_2$.

in **Sys** whenever there is a translation (in the opposite direction), there is indeed an arrow $S_1 \sqcup S \to S_2 \sqcup S$.

In the case of the composition, however, this would require a translation $\mathcal{L}(S_2 \diamond S) \to \mathcal{L}(S_1 \diamond S)$, and this cannot be done in general since we have no *prima facie* information about the set of function symbols $F_{\mathcal{L}(S_2\diamond S)}(\Sigma, \mathcal{R})$. However, if we restrict the arrows in **Sys** to be those associated with sub-systems, combination of systems, and compositions of such arrows, then it is easy to see that the required translations exist (the proof of this makes essential use of (2.16)).

If we make this restriction of arrows, then the axioms (2.14), (2.17–2.20), mean that, essentially, **Sys** has the structure of a *symmetric monoidal*[11] category in which the monoidal product operation is '$\diamond$', and the left and right unit object is 1. There is also a monoidal structure associated with the disjoint sum '$\sqcup$', with 0 as the unit object.

We say 'essentially' because in order to comply with all the axioms of a monoidal category, **Sys** must satisfy certain additional, so-called, 'coherence' axioms. However, from a physical perspective these are very plausible statements about (i) how the unit object 1 intertwines with the $\diamond$-operation; how the null object intertwines with the $\sqcup$-operation; and (iii) certain properties of quadruple products (and disjoint sums) of systems.

A simple example of a category Sys. It might be helpful at this point to give a simple example of a category **Sys**. To that end, let S denote a point particle that moves in three dimensions, and let us suppose that S has no sub-systems other than the trivial system 1. Then $S \diamond S$ is defined to be a pair of particles moving in three dimensions, and so on. Thus the objects in our category are 1, S, $S\diamond S$, ..., $S\diamond S\diamond \cdots S$ ... where the '$\diamond$' operation is formed any finite number of times.

At this stage, the only arrows are those that are associated with the constituents of a composite system. However, we could contemplate adding to the systems the disjoint sum $S \sqcup (S \diamond S)$ which is a system that is either one particle or two particles (but, of course, not both at the same time). And, clearly, we could extend this to $S \sqcup (S \diamond S) \sqcup (S \diamond S \diamond S)$, and so on. Each of these disjoint sums comes with its own arrows, as explained above.

Note that this particular category of systems has the property that it can be treated using either classical physics or quantum theory.

[11] In the actual definition of a monoidal category the two isomorphisms in (2.17) are separated from each other, whereas we have identified them. Further more, these isomorphism are required to be natural. This seems a correct thing to require in our case, too.

2.3 Representations of Sys in Topoi

We assume that all the systems in **Sys** are to be treated with the same theory type. We also assume that systems in **Sys** with the *same* language are to be represented in the same topos. Then we define:[12]

Definition 2.1 *A* topos realisation *of* **Sys** *is an association, ϕ, to each system S in* **Sys***, of a triple $\phi(S) = \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_\phi(S)\rangle$ where:*

(i) $\tau_\phi(S)$ is the topos in which the theory-type applied to system S is to be realised.

(ii) $\mathcal{L}(S)$ is the local language in **Loc** *that is associated with S. This is not dependent on the realisation ϕ.*

(iii) $\rho_{\phi,S}$ is a representation of the local language $\mathcal{L}(S)$ in the topos $\tau_\phi(S)$. As a more descriptive piece of notation we write $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$. The key part of this representation is the map

$$\rho_{\phi,S} : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \tag{2.21}$$

where $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$ are the state object and quantity-value object, respectively, of the representation ϕ in the topos $\tau_\phi(S)$. As a convenient piece of notation we write $A_{\phi,S} := \rho_{\phi,S}(A)$ for all $A \in F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$.

This definition is only partial; the possibility of extending it will be discussed shortly.

Now, if $j : S_1 \to S$ is an arrow in **Sys**, then there is a translation arrow $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$. Thus we have the beginnings of a commutative diagram

$$\begin{tikzcd} S_1 \arrow[r, "\phi"] \arrow[d, "j"'] & \langle \rho_{\phi,S_1}, \mathcal{L}(S_1), \tau_\phi(S_1)\rangle \\ S \arrow[r, "\phi"'] & \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_\phi(S)\rangle \arrow[u, "? \times \mathcal{L}(j) \times ?"'] \end{tikzcd} \tag{2.22}$$

However, to be useful, the arrow on the right hand side of this diagram should refer to some relation between (i) the topoi $\tau_\phi(S_1)$ and $\tau_\phi(S)$; and (ii) the realisations $\rho_{\phi,S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_\phi(S_1)$ and $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$: this is the significance of the two '?' symbols in the arrow written '$? \times \mathcal{L}(j) \times ?$'.

Indeed, as things stand, Definition 2.1 says nothing about relations between the topoi representations of different systems in **Sys**. We are particularly interested in the

[12]As emphasised already, the association $S \mapsto \mathcal{L}(S)$ is generally not one-to-one: i.e., many systems may share the same language. Thus, when we come discuss the representation of the language $\mathcal{L}(S)$ in a topos, the extra information about the system S is used in fixing the representation.

situation where there are two different systems S_1 and S with an arrow $j : S_1 \to S$ in **Sys**.

We know that the arrow j is associated with a translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$, and an attractive possibility, therefore, would be to seek, or postulate, a 'covering' map $\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$ to be construed as a topos representation of the translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$, and hence of the arrow $j : S_1 \to S$ in **Sys**.

This raises the questions of what properties these 'translation representations' should possess in order to justify saying that they 'cover' the translations. A minimal requirement is that if $k : S_2 \to S_1$ and $j : S_1 \to S$, then the map $\phi(\mathcal{L}(j \circ k)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_2)}\big(\Sigma_{\phi,S_2}, \mathcal{R}_{\phi,S_2}\big)$ factorises as

$$\phi(\mathcal{L}(j \circ k)) = \phi(\mathcal{L}(k)) \circ \phi(\mathcal{L}(j)). \tag{2.23}$$

We also require that

$$\phi(\mathcal{L}(\mathrm{id}_S)) = \mathrm{id} : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \tag{2.24}$$

for all systems S.

The conditions (2.23) and (2.24) seem eminently plausible, and they are not particularly strong. A far more restrictive axiom would be to require the following diagram to commute:

$$\begin{tikzcd}
F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S}"] \arrow[d, "\mathcal{L}(j)"'] & \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \arrow[d, "\phi(\mathcal{L}(j))"] \\
F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S_1}"'] & \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)
\end{tikzcd} \tag{2.25}$$

At first sight, this requirement seems very appealing. However, caution is needed when postulating 'axioms' for a theoretical structure in physics. It is easy to get captivated by the underlying mathematics and to assume, erroneously, that what is mathematically elegant is necessarily true in the physical theory.

The translation $\phi(\mathcal{L}(j))$ maps an arrow from $\Sigma_{\phi,S}$ to $\mathcal{R}_{\phi,S}$ to an arrow from Σ_{ϕ,S_1} to $\mathcal{R}_{\phi,S_1}$. Intuitively, if Σ_{ϕ,S_1} is a 'much larger' object than $\Sigma_{\phi,S}$ (they lie in different topoi, so no direct comparison is available), the translation can only be 'faithful' on some part of Σ_{ϕ,S_1} that can be identified with (the 'image' of) $\Sigma_{\phi,S}$. A concrete example of this will show up in the treatment of composite quantum systems, see Subsection 4.3. As one might expect, a form of entanglement plays a role here.

2.4 Classical Physics in This Form

2.4.1 The Rules so Far.

Constructing maps $\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$ is likely to be complicated when $\tau_\phi(S)$ and $\tau_\phi(S_1)$ are different topoi, and so we begin with the example of classical physics, where the topos is always **Sets**.

In general, we are interested in the relation(s) between the representations $\rho_{\phi,S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_\phi(S_1)$ and $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$ that is associated with an arrow $j : S_1 \to S$ in **Sys**. In classical physics, we only have to study the relation between the representations $\rho_{\sigma,S_1} : \mathcal{L}(S_1) \rightsquigarrow$ **Sets** and $\rho_{\sigma,S} : \mathcal{L}(S) \rightsquigarrow$ **Sets**.

Let us summarise what we have said so far (with σ denoting the **Sets**-realisation of classical physics):

1. For any system S in **Sys**, a representation $\rho_{\sigma,S} : \mathcal{L}(S) \rightsquigarrow$ **Sets** consists of the following ingredients.

   (a) The ground symbol Σ is represented by a symplectic manifold, $\Sigma_{\sigma,S} := \rho_{\sigma,S}(\Sigma)$, that serves as the classical state space.

   (b) For all systems S, the ground symbol $\mathcal{R}$ is represented by the real numbers $\mathbb{R}$, i.e., $\mathcal{R}_{\sigma,S} = \mathbb{R}$, where $\mathcal{R}_{\sigma,S} := \rho_{\sigma,S}(\mathcal{R})$.

   (c) Each function symbol $A : \Sigma \to \mathcal{R}$ in $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ is represented by a function $A_{\sigma,S} = \rho_{\sigma,S}(A) : \Sigma_{\sigma,S} \to \mathbb{R}$ in the set of functions[13] $C(\Sigma_{\sigma,S}; \mathbb{R})$.

2. The trivial system is mapped to a singleton set $\{*\}$ (viewed as a zero-dimensional symplectic manifold):
$$\Sigma_{\sigma,1} := \{*\}. \tag{2.26}$$
   The empty system is represented by the empty set:
$$\Sigma_{\sigma,0} := \emptyset. \tag{2.27}$$

3. Propositions about the system S are represented by (Borel) subsets of the state space $\Sigma_{\sigma,S}$.

4. The composite system $S_1 \diamond S_2$ is represented by the Cartesian product $\Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$; i.e.,
$$\Sigma_{\sigma,\, S_1 \diamond S_2} \simeq \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}. \tag{2.28}$$
   The disjoint sum $S_1 \sqcup S_2$ is represented by the disjoint union $\Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_2}$;i.e.,
$$\Sigma_{\sigma,S_1 \sqcup S_2} \simeq \Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_2}. \tag{2.29}$$

[13] In practice, these functions are required to be measurable with respect to the Borel structures on the symplectic manifold Σ_σ and $\mathbb{R}$. Many of the functions will also be smooth, but we will not go into such details here.

5. Let $j : S_1 \to S$ be an arrow in **Sys**. Then

   (a) There is a translation map $\mathcal{L}(j) : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$.

   (b) There is a symplectic function $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$ from the symplectic manifold Σ_{σ,S_1} to the symplectic manifold $\Sigma_{\sigma,S}$.

The existence of this function $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$ follows directly from the properties of sub-systems and composite systems in classical physics. It is discussed in detail below in Section (2.4.2). As we shall see, it underpins the classical realisation of our axioms.

These properties of the arrows stem from the fact that the linguistic function symbols in $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ are represented by real-valued functions in $C(\Sigma_{\sigma,S}, \mathbb{R})$. Thus we can write $\rho_{\sigma,S} : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to C(\Sigma_{\sigma,S}, \mathbb{R})$, and similarly $\rho_{\sigma,S_1} : F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \to C(\Sigma_{\sigma,S_1}, \mathbb{R})$. The diagram in (2.25) now becomes

$$\begin{tikzcd} F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\sigma,S}"] \arrow[d, "\mathcal{L}(j)"'] & C(\Sigma_{\sigma,S}, \mathbb{R}) \arrow[d, "\sigma(\mathcal{L}(j))"] \\ F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\sigma,S_1}"'] & C(\Sigma_{\sigma,S_1}, \mathbb{R}) \end{tikzcd} \tag{2.30}$$

and, therefore, the question of interest is if there is a 'translation representation' function $\sigma(\mathcal{L}(j)) : C(\Sigma_{\sigma,S}, \mathbb{R}) \to C(\Sigma_{\sigma,S_1}, \mathbb{R})$ so that this diagram commutes.

Now, as stated above, a physical quantity, A, for the system S is represented in classical physics by a real-valued function $A_{\sigma,S} = \rho_{\sigma,S}(A) : \Sigma_{\sigma,S} \to \mathbb{R}$. Similarly, the representation of $\mathcal{L}(j)(A)$ for S_1 is given by a function $A_{\sigma,S_1} := \rho_{\sigma,S_1}(A) : \Sigma_{\sigma,S_1} \to \mathbb{R}$. However, in this classical case we also have the function $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$, and it is clear that we can use it to define $[\rho_{\sigma,S_1}(\mathcal{L}(j)(A)](s) := \rho_{\sigma,S}(A)\big(\sigma(j)(s)\big)$ for all $s \in \Sigma_{\sigma,S_1}$. In other words

$$\rho_{\sigma,S_1}\big(\mathcal{L}(j)(A)\big) = \rho_{\sigma,S}(A) \circ \sigma(j) \tag{2.31}$$

or, in simpler notation

$$\big((\mathcal{L}(j)(A)\big)_{\sigma,S_1} = A_{\sigma,S} \circ \sigma(j). \tag{2.32}$$

But then it is clear that a translation-representation function $\sigma(\mathcal{L}(j)) : C(\Sigma_{\sigma,S}, \mathbb{R}) \to C(\Sigma_{\sigma,S_1}, \mathbb{R})$ with the desired property of making (2.30) commute can be defined by

$$\sigma(\mathcal{L}(j))(f) := f \circ \sigma(j) \tag{2.33}$$

for all $f \in C(\Sigma_{\sigma,S}, \mathbb{R})$; i.e., the function $\sigma(\mathcal{L}(j))(f) : \Sigma_{\sigma,S_1} \to \mathbb{R}$ is the usual pull-back of the function $f : \Sigma_{\sigma,S} \to \mathbb{R}$ by the function $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$. Thus, in the case of classical physics, the commutative diagram in (2.22) can be completed to give

$$\begin{tikzcd} S_1 \arrow[r, "\sigma"] \arrow[d, "j"'] & \langle \rho_{\sigma,S_1}, \mathcal{L}(S_1), \mathbf{Sets}\rangle \\ S \arrow[r, "\sigma"'] & \langle \rho_{\sigma,S}, \mathcal{L}(S), \mathbf{Sets}\rangle \arrow[u, "\sigma(\mathcal{L}(j)) \times \mathcal{L}(j) \times \mathrm{id}"'] \end{tikzcd} \tag{2.34}$$

2.4.2 Details of the Translation Representation.

The translation representation for a disjoint sum of classical systems. We first consider arrows of the form

$$S_1 \xrightarrow{i_1} S_1 \sqcup S_2 \xleftarrow{i_2} S_2 \tag{2.35}$$

from the components S_1, S_2 to the disjoint sum $S_1 \sqcup S_2$. The systems S_1, S_2 and $S_1 \sqcup S_2$ have symplectic manifolds Σ_{σ,S_1}, Σ_{σ,S_2} and $\Sigma_{\sigma,S_1\sqcup S_2} = \Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_2}$. We write $i := i_1$.

Let S be a classical system. We assume that the function symbols $A \in F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ in the language $\mathcal{L}(S)$ are in bijective correspondence with an appropriate subset of the functions $A_{\sigma,S} \in C(\Sigma_{\sigma,S}, \mathbb{R})$.[14]

There is an obvious translation representation. For if $A \in F_{\mathcal{L}(S_1\sqcup S_2)}\big(\Sigma, \mathcal{R}\big)$, then since $\Sigma_{\sigma,S_1\sqcup S_2} = \Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_1}$, the associated function $A_{\sigma,S_1\sqcup S_2} : \Sigma_{\sigma,S_1\sqcup S_2} \rightarrow \mathbb{R}$ is given by a pair of functions $A_1 \in C(\Sigma_{\sigma,S_1}, \mathbb{R})$ and $A_2 \in C(\Sigma_{\sigma,S_2}, \mathbb{R})$; we write $A_{\sigma,S_1\sqcup S_2} = \langle A_1, A_2\rangle$. It is natural to demand that the translation representation $\sigma(\mathcal{L}(i))(A_{\sigma,S_1\sqcup S_2})$ is A_1. Note that what is essentially being discussed here is the classical-physics representation of the relation (2.1).

The canonical choice for $\sigma(i)$ is

$$\begin{aligned} \sigma(i) : \Sigma_{\sigma,S_1} &\rightarrow \Sigma_{\sigma,S_1\sqcup S_2} = \Sigma_{\sigma,S_1} \textstyle\coprod \Sigma_{\sigma,S_2} && (2.36)\\ s_1 &\mapsto s_1. && (2.37) \end{aligned}$$

Then the pull-back along $\sigma(i)$,

$$\begin{aligned} \sigma(i)^* : C(\Sigma_{\sigma,S_1\sqcup S_2}, \mathbb{R}) &\rightarrow C(\Sigma_{\sigma,S_1}, \mathbb{R}) && (2.38)\\ A_{\sigma,S_1\sqcup S_2} &\mapsto A_{\sigma,S_1\sqcup S_2} \circ \sigma(i), && (2.39) \end{aligned}$$

maps (or 'translates') the topos representative $A_{\sigma,S_1\sqcup S_2} = \langle A_1, A_2\rangle$ of the function symbol $A \in F_{\mathcal{L}(S_1\sqcup S_2)}\big(\Sigma, \mathcal{R}\big)$ to a real-valued function $A_{\sigma,S_1\sqcup S_2} \circ \sigma(i)$ on Σ_{σ,S_1}. This function is clearly equal to A_1.

[14] Depending on the setting, one can assume that $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ contains function symbols corresponding bijectively to measurable, continuous or smooth functions.

The translation in the case of a composite classical system. We now consider arrows in **Sys** of the form

$$S_1 \overset{p_1}{\leftarrow} S_1 \diamond S_2 \overset{p_2}{\rightarrow} S_2 \tag{2.40}$$

from the composite classical system $S_1 \diamond S_2$ to the constituent systems S_1 and S_2. Here, p_1 signals that S_1 is a constituent of the composite system $S_1 \diamond S_2$, likewise p_2. The systems S_1, S_2 and $S_1 \diamond S_2$ have symplectic manifolds Σ_{σ,S_1}, Σ_{σ,S_2} and $\Sigma_{\sigma,S_1\diamond S_2} = \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$, respectively; i.e., the state space of the composite system $S_1 \diamond S_2$ is the cartesian product of the state spaces of the components. For typographical simplicity in what follows we denote $p := p_1$.

There is a canonical translation $\mathcal{L}(p)$ between the languages $\mathcal{L}(S_1)$ and $\mathcal{L}(S_1 \diamond S_2)$ whose representation is the following. Namely, if A is in $F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$, then the corresponding function $A_{\sigma,S_1} \in C(\Sigma_{\sigma,S_1}, \mathbb{R})$ is translated to a function $\sigma(\mathcal{L}(p))(A_{\sigma,S_1}) \in C(\Sigma_{\sigma,S_1\diamond S_2}, \mathbb{R})$ such that

$$\sigma(\mathcal{L}(p))(A_{\sigma,S_1})(s_1, s_2) = A_{\sigma,S_1}(s_1) \tag{2.41}$$

for all $(s_1, s_2) \in \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$.

This natural translation representation is based on the fact that, for the symplectic manifold $\Sigma_{\sigma,S_1\diamond S_2} = \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$, each point $s \in \Sigma_{\sigma,S_1\diamond S_2}$ can be identified with a pair, (s_1, s_2), of points $s_1 \in \Sigma_{\sigma,S_1}$ and $s_2 \in \Sigma_{\sigma,S_2}$. This is possible since the cartesian product $\Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$ is a product in the categorial sense and hence has projections $\Sigma_{\sigma,S_1} \leftarrow \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2} \rightarrow \Sigma_{\sigma,S_2}$. Then the translation representation of functions is constructed in a straightforward manner. Thus, let

$$\begin{aligned} \sigma(p) : \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2} &\rightarrow \Sigma_{\sigma,S_1} \\ (s_1, s_2) &\mapsto s_1 \end{aligned} \tag{2.42}$$

be the canonical projection. Then, if $A_{\sigma,S_1} \in C(\Sigma_{\sigma,S_1}, \mathbb{R})$, the function

$$A_{\sigma,S_1} \circ \sigma(p) \in C(\Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}, \mathbb{R}) \tag{2.43}$$

is such that, for all $(s_1, s_2) \in \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$,

$$A_{\sigma,S_1} \circ \sigma(p)(s_1, s_2) = A_{\sigma,S_1}(s_1). \tag{2.44}$$

Thus we can define

$$\sigma(\mathcal{L}(p))(A_{\sigma,S_1}) := A_{\sigma,S_1} \circ \sigma(p). \tag{2.45}$$

Clearly, $\sigma(\mathcal{L}(p))(A_{\sigma,S_1})$ can be seen as the representation of the function symbol $A \diamond 1 \in F_{\mathcal{L}(S_1\diamond S_2)}\big(\Sigma, \mathcal{R}\big)$.

3 Theories of Physics in a General Topos

3.1 The Pull-Back Operations

3.1.1 The Pull-Back of Physical Quantities.

Motivated by the above, let us try now to see what can be said about the scheme in general. Basically, what is involved is the topos representation of translations of languages. To be more precise, let $j : S_1 \to S$ be an arrow in **Sys**, so that there is a translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$ defined by the translation function $\mathcal{L}(j) : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$. Now suppose that the systems S and S_1 are represented in the topoi $\tau_\phi(S)$ and $\tau_\phi(S_1)$ respectively. Then, in these representations, the function symbols of signature $\Sigma \to \mathcal{R}$ in $\mathcal{L}(S)$ and $\mathcal{L}(S_1)$ are represented by elements of $\mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big)$ and $\mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$ respectively.

Our task is to find a function

$$\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big) \tag{3.1}$$

that can be construed as the topos representation of the translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$, and hence of the arrow $j : S_1 \to S$ in **Sys**. We are particularly interested in seeing if $\phi(\mathcal{L}(j))$ can be chosen so that the following diagram, (see (2.25)) commutes:

$$\begin{tikzcd}
F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S}"] \arrow[d, "\mathcal{L}(j)"'] & \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \arrow[d, "\phi(\mathcal{L}(j))"] \\
F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S_1}"'] & \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)
\end{tikzcd} \tag{3.2}$$

However, as has been emphasised already, it is not clear that one *should* expect to find a function $\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$ with this property. The existence and/or properties of such a function will be dependent on the theory-type, and it seems unlikely that much can be said in general about the diagram (3.2). Nevertheless, let us see how far we *can* get in discussing the existence of such a function in general.

Thus, if $\mu \in \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big)$, the critical question is if there is some 'natural' way whereby this arrow can be 'pulled-back' to give an element $\phi(\mathcal{L}(j))(\mu) \in \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$.

The first pertinent remark is that μ is an arrow in the topos $\tau_\phi(S)$, whereas the sought-for pull-back will be an arrow in the topos $\tau_\phi(S_1)$, and so we need a mechanism for getting from one topos to the other (this problem, of course, does not arise in classical physics since the topos of every representation is always **Sets**).

The obvious way of implementing this change of topos is via some *functor*, $\tau_\phi(j)$ from $\tau_\phi(S)$ to $\tau_\phi(S_1)$. Indeed, given such a functor, an arrow $\mu : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ in $\tau_\phi(S)$ is transformed to the arrow

$$\tau_\phi(j)(\mu) : \tau_\phi(j)(\Sigma_{\phi,S}) \to \tau_\phi(j)(\mathcal{R}_{\phi,S}) \tag{3.3}$$

in $\tau_\phi(S_1)$.

To convert this to an arrow from Σ_{ϕ,S_1} to $\mathcal{R}_{\phi,S_1}$, we need to supplement (3.3) with a pair of arrows $\phi(j), \beta_\phi(j)$ in $\tau_\phi(S_1)$ to get the diagram:

$$\begin{tikzcd}
\tau_\phi(j)(\Sigma_{\phi,S}) \arrow[r, "\tau_\phi(j)(\mu)"] & \tau_\phi(j)(\mathcal{R}_{\phi,S}) \arrow[d, "\beta_\phi(j)"] \\
\Sigma_{\phi,S_1} \arrow[u, "\phi(j)"] & \mathcal{R}_{\phi,S_1}
\end{tikzcd} \tag{3.4}$$

The pull-back, $\phi(\mathcal{L}(j))(\mu) \in \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$, with respect to these choices can then be defined as

$$\phi(\mathcal{L}(j))(\mu) := \beta_\phi(j) \circ \tau_\phi(j)(\mu) \circ \phi(j). \tag{3.5}$$

It follows that a key part of the construction of a topos representation, ϕ, of **Sys** will be to specify the functor $\tau_\phi(j)$ from $\tau_\phi(S)$ to $\tau_\phi(S_1)$, and the arrows $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ and $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi,S}) \to \mathcal{R}_{\phi,S_1}$ in the topos $\tau_\phi(S_1)$. These need to be defined in such a way as to be consistent with a chain of arrows $S_2 \to S_1 \to S$.

When applied to the representative $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ of a physical quantity $A \in F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$, the diagram (3.4) becomes (augmented with the upper half)

$$\begin{tikzcd}
\Sigma_{\phi,S} \arrow[r, "A_{\phi,S}"] \arrow[d, "\tau_\phi(j)"'] & \mathcal{R}_{\phi,S} \arrow[d, "\tau_\phi(j)"] \\
\tau_\phi(j)(\Sigma_{\phi,S}) \arrow[r, "\tau_\phi(j)(A_{\phi,S})"'] & \tau_\phi(j)(\mathcal{R}_{\phi,S}) \arrow[d, "\beta_\phi(j)"] \\
\Sigma_{\phi,S_1} \arrow[u, "\phi(j)"] \arrow[r, "\phi(\mathcal{L}(j))(A_{\phi,S})"'] & \mathcal{R}_{\phi,S_1}
\end{tikzcd} \tag{3.6}$$

The commutativity of (3.2) would then require

$$\phi(\mathcal{L}(j))(A_{\phi,S}) = (\mathcal{L}(j)A)_{\phi,S_1} \tag{3.7}$$

or, in a more expanded notation,

$$\phi(\mathcal{L}(j)) \circ \rho_{\phi,S} = \rho_{\phi,S_1} \circ \mathcal{L}(j), \tag{3.8}$$

where both the left hand side and the right hand side of (3.8) are mappings from $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ to $\mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$.

Note that the analogous diagram in classical physics is simply

$$
\begin{tikzcd}
\Sigma_{\sigma,S} \arrow[r, "A_{\sigma,S}"] & \mathbb{R} \arrow[d, "\mathrm{id}"] \\
\Sigma_{\sigma,S_1} \arrow[u, "\sigma(j)"] \arrow[r, "\sigma(\mathcal{L}(j))(A_{\sigma,S})"'] & \mathbb{R}
\end{tikzcd}
\tag{3.9}
$$

and the commutativity/pull-back condition (3.7) becomes

$$\sigma(\mathcal{L}(j))(A_{\sigma,S}) = (\mathcal{L}(j)A)_{\phi,S_1} \tag{3.10}$$

which is satisfied by virtue of (2.33).

It is clear from the above that the arrow $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ can be viewed as the topos analogue of the map $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$ that arises in classical physics whenever there is an arrow $j : S_1 \to S$.

3.1.2 The Pull-Back of Propositions.

More insight can be gained into the nature of the triple $\langle \tau_\phi(j), \phi(j), \beta_\phi(j)\rangle$ by considering the analogous operation for propositions. First, consider an arrow $j : S_1 \to S$ in **Sys** in classical physics. Associated with this there is (i) a translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$; (ii) an associated translation mapping $\mathcal{L}(j) : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$; and (iii) a symplectic function $\sigma(j) : \Sigma_{\sigma,S_1} \to \Sigma_{\sigma,S}$.

Let K be a (Borel) subset of the state space, $\Sigma_{\sigma,S}$; hence K represents a proposition about the system S. Then $\sigma(j)^*(K) := \sigma(j)^{-1}(K)$ is a subset of Σ_{σ,S_1} and, as such, represents a proposition about the system S_1. We say that $\sigma(j)^*(K)$ is the *pull-back* to Σ_{σ,S_1} of the S-proposition represented by K. The existence of such pull-backs is part of the consistency of the representation of propositions in classical mechanics, and it is important to understand what the analogue of this is in our topos scheme.

Consider the general case with the two systems S_1, S as above. Then let K be a proposition, represented as a sub-object of $\Sigma_{\phi,S}$, with a monic arrow $i_K : K \hookrightarrow \Sigma_{\phi,S}$. The question now is if the triple $\langle \tau_\phi(j), \phi(j), \beta_\phi(j)\rangle$ can be used to pull K back to give a proposition in $\tau(S_1)$, i.e., a sub-object of Σ_{ϕ,S_1}?

The first requirement is that the functor $\tau_\phi(j) : \tau_\phi(S) \to \tau_\phi(S_1)$ should *preserve monics*; for example by being left-exact. In this case, the monic arrow $i_K : K \hookrightarrow \Sigma_{\phi,S}$ in $\tau_\phi(S)$ is transformed to the monic arrow

$$\tau_\phi(j)(i_K) : \tau_\phi(j)(K) \hookrightarrow \tau_\phi(j)(\Sigma_{\phi,S}) \tag{3.11}$$

in $\tau_\phi(S_1)$; thus $\tau_\phi(j)(K)$ is a sub-object of $\tau_\phi(j)(\Sigma_{\phi,S})$ in $\tau_\phi(S_1)$. It is a property of a topos that the pull-back of a monic arrow is monic ; i.e., if $M \hookrightarrow Y$ is monic, and if $\psi : X \to Y$, then $\psi^{-1}(M)$ is a sub-object of X. Therefore, in the case of interest, the monic arrow $\tau_\phi(j)(i_K) : \tau_\phi(j)(K) \hookrightarrow \tau_\phi(j)(\Sigma_{\phi,S})$ can be pulled back along $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ (see diagram (3.6)) to give the monic $\phi(j)^{-1}(\tau_\phi(j)(K)) \subseteq \Sigma_{\phi,S_1}$. This is a candidate for the pull-back of the proposition represented by the sub-object $K \subseteq \Sigma_{\phi,S}$.

In conclusion, propositions can be pulled-back provided that the functor $\tau_\phi(j) : \tau_\phi(S) \to \tau_\phi(S_1)$ preserves monics; for example, by being left-exact. In fact, the property of being left-exact is so natural that we shall add it to our list of postulates; see below.

3.2 The Topos Rules for Theories of Physics

We will now present our general rules for using topos theory in the mathematical representation of physical systems and their theories.

Definition 3.1 *The category $\mathcal{M}(\mathbf{Sys})$ is defined as follows:*

1. *The objects of $\mathcal{M}(\mathbf{Sys})$ are the topoi that are to be used in representing the systems in* $\mathbf{Sys}$.
2. *The arrows from τ_1 to τ_2 are defined to be the left-exact functors from τ_1 to τ_2.*

Definition 3.2 *The rules for using topos theory are as follows:*

1. A *topos realisation*, ϕ, of $\mathbf{Sys}$ in $\mathcal{M}(\mathbf{Sys})$ is an assignment, to each system S in $\mathbf{Sys}$, of a triple $\phi(S) = \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_\phi(S) \rangle$ where:
   (a) $\tau_\phi(S)$ is the topos in $\mathcal{M}(\mathbf{Sys})$ in which the theory-type applied to system S is to be realised.
   (b) $\mathcal{L}(S)$ is the local language that is associated with S. This is independent of the realisation, ϕ, of $\mathbf{Sys}$ in $\mathcal{M}(\mathbf{Sys})$.
   (c) $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$ is a representation of the local language $\mathcal{L}(S)$ in the topos $\tau_\phi(S)$.
   (d) In addition, for each arrow $j : S_1 \to S$ in $\mathbf{Sys}$ there is a triple $\langle \tau_\phi(j), \phi(j), \beta_\phi(j) \rangle$ that interpolates between $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$ and $\rho_{\phi,S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_\phi(S_1)$; for details see below.
2. (a) The representations, $\rho_{\phi,S}(\Sigma)$ and $\rho_{\phi,S}(\mathcal{R})$, of the ground symbols Σ and $\mathcal{R}$ in $\mathcal{L}(S)$ are denoted $\Sigma_{\phi,S}$ and $\mathcal{R}_{\phi,S}$, respectively. They are known as the 'state object' and 'quantity-value object' in $\tau_\phi(S)$.
   (b) The representation by $\rho_{\phi,S}$ of each function symbol $A : \Sigma \to \mathcal{R}$ of the system S is an arrow, $\rho_{\phi,S}(A) : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ in $\tau_\phi(S)$; we will usually denote this arrow as $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$.

(c) Propositions about the system S are represented by sub-objects of $\Sigma_{\phi,S}$. These will typically be of the form $A^{-1}_{\phi,S}(\Xi)$, where Ξ is a sub-object of $\mathcal{R}_{\phi,S}$.[15]

3. Generally, there are no 'microstates' for the system S; i.e., no global elements (arrows $1 \to \Sigma_{\phi,S}$) of the state object $\Sigma_{\phi,S}$; or, if there are any, they may not be enough to determine $\Sigma_{\phi,S}$ as an object in $\tau_\phi(S)$.

   Instead, the role of a state is played by a 'truth sub-object' $\mathbb{T}$ of $P\Sigma_{\phi,S}$.[16] If $\gamma \in \mathrm{Sub}(\Sigma_{\phi,S}) \simeq \Gamma(P\Sigma_{\phi,S})$, the 'truth of the proposition represented by γ' is defined to be
$$\nu(\ulcorner\gamma\urcorner \in \mathbb{T}) = [\![\, \tilde{\gamma} \in \tilde{\mathbb{T}}\,]\!]_\phi \,\langle \gamma, \mathbb{T}\rangle \tag{3.12}$$
   See paper II for full information on the idea of a 'truth object' [2].

4. There is a 'unit object' $1_{\mathcal{M}(\mathbf{Sys})}$ in $\mathcal{M}(\mathbf{Sys})$ such that if $1_{\mathbf{Sys}}$ denotes the trivial system in $\mathbf{Sys}$ then, for all topos realisations ϕ,
$$\tau_\phi(1_{\mathbf{Sys}}) = 1_{\mathcal{M}(\mathbf{Sys})}. \tag{3.13}$$
   Motivated by the results for quantum theory (see Section 4.2), we postulate that the unit object $1_{\mathcal{M}(\mathbf{Sys})}$ in $\mathcal{M}(\mathbf{Sys})$ is the category of sets:
$$1_{\mathcal{M}(\mathbf{Sys})} = \mathbf{Sets}. \tag{3.14}$$

5. To each arrow $j : S_1 \to S$ in $\mathbf{Sys}$, we have the following:

   (a) There is a translation $\mathcal{L}(j) : \mathcal{L}(S) \to \mathcal{L}(S_1)$. This is specified by a map between function symbols: $\mathcal{L}(j) : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$.

   (b) With the translation $\mathcal{L}(j) : F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ there is associated a corresponding function
$$\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big). \tag{3.15}$$
   These may, or may not, fit together in the commutative diagram:
$$\begin{tikzcd} F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S}"] \arrow[d, "\mathcal{L}(j)"'] & \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \arrow[d, "\phi(\mathcal{L}(j))"] \\ F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \arrow[r, "\rho_{\phi,S_1}"'] & \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big) \end{tikzcd} \tag{3.16}$$

---

[15] Here, $A^{-1}_{\phi,S}(\Xi)$ denotes the sub-object of $\Sigma_{\phi,S}$ whose characteristic arrow is $\chi_\Xi \circ A_{\phi,S} : \Sigma_{\phi,S} \to \Omega_{\tau_\phi(S)}$, where $\chi_\Xi : \mathcal{R}_{\phi,S} \to \Omega_{\tau_\phi(S)}$ is the characteristic arrow of the sub-object Ξ.

[16] In classical physics, the truth object corresponding to a microstate s is the collection of all propositions that are true in the state s.

(c) The function $\phi(\mathcal{L}(j)) : \mathrm{Hom}_{\tau_\phi(S)}\big(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}\big) \to \mathrm{Hom}_{\tau_\phi(S_1)}\big(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}\big)$ is built from the following ingredients. For each topos realisation ϕ, there is a triple $\langle \tau_\phi(j), \phi(j), \beta_\phi(j)\rangle$ where:

i. $\tau_\phi(j) : \tau_\phi(S) \to \tau_\phi(S_1)$ is a left-exact functor; i.e., an arrow in $\mathcal{M}(\mathbf{Sys})$.

ii. $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)\big(\Sigma_{\phi,S}\big)$ is an arrow in $\tau_\phi(S_1)$.

iii. $\beta_\phi(j) : \tau_\phi(j)\big(\mathcal{R}_{\phi,S}\big) \to \mathcal{R}_{\phi,S_1}$ is an arrow in $\tau_\phi(S_1)$.

These fit together in the diagram

$$\begin{tikzcd}
\Sigma_{\phi,S} \arrow[r, "A_{\phi,S}"] \arrow[d, "\tau_\phi(j)"'] & \mathcal{R}_{\phi,S} \arrow[d, "\tau_\phi(j)"] \\
\tau_\phi(j)(\Sigma_{\phi,S}) \arrow[r, "\tau_\phi(j)(A_{\phi,S})"'] & \tau_\phi(j)(\mathcal{R}_{\phi,S}) \arrow[d, "\beta_\phi(j)"] \\
\Sigma_{\phi,S_1} \arrow[u, "\phi(j)"] \arrow[r, "\phi(\mathcal{L}(j))(A_{\phi,S})"'] & \mathcal{R}_{\phi,S_1}
\end{tikzcd} \tag{3.17}$$

The arrows $\phi(j)$ and $\beta_\phi(j)$ should behave appropriately under composition of arrows in **Sys**.

The commutativity of the diagram (3.16) is equivalent to the relation

$$\phi(\mathcal{L}(j))(A_{\phi,S}) = [\mathcal{L}(j)(A)]_{\phi,S_1} \tag{3.18}$$

for all $A \in F_{\mathcal{L}(\phi,S)}\big(\Sigma, \mathcal{R}\big)$. As we keep emphasising, the satisfaction or otherwise of this relation will depend on the theory-type and, possibly, the representation ϕ.

(d) If a proposition in $\tau_\phi(S)$ is represented by the monic arrow, $K \hookrightarrow \Sigma_{\phi,S}$, the 'pull-back' of this proposition to $\tau_\phi(S_1)$ is defined to be $\phi(j)^{-1}\big(\tau_\phi(j)(K)\big) \subseteq \Sigma_{\phi,S_1}$.

6. (a) If S_1 is a sub-system of S, with an associated arrow $i : S_1 \to S$ in **Sys** then, in the diagram in (3.17), the arrow $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ is a monic arrow in $\tau_\phi(S_1)$.

In other words, Σ_{ϕ,S_1} is a sub-object of $\tau_\phi(j)(\Sigma_{\phi,S})$, which is denoted

$$\Sigma_{\phi,S_1} \subseteq \tau_\phi(j)(\Sigma_{\phi,S}). \tag{3.19}$$

We may also want to conjecture

$$\mathcal{R}_{\phi,S_1} \simeq \tau_\phi(j)\big(\mathcal{R}_{\phi,S}\big). \tag{3.20}$$

(b) Another possible conjecture is the following: if $j : S_1 \to S$ is an epic arrow in **Sys**, then, in the diagram in (3.17), the arrow $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ is an epic arrow in $\tau_\phi(S_1)$.

In particular, for the epic arrow $p_1 : S_1 \diamond S_2 \to S_1$, the arrow $\phi(p_1) : \Sigma_{\phi,S_1\diamond S_2} \to \tau_\phi\big(\Sigma_{\phi,S_1}\big)$ is an epic arrow in the topos $\tau_\phi(S_1 \diamond S_2)$.

One should not read Rule 2. above as implying that the choice of the state object and quantity-value object are *unique* for any give system S. These objects would at best be selected only up to isomorphism in the topos $\tau(S)$. Such morphisms in the topos $\tau(S)$[17] can be expected to play a key role in developing the topos analogue of the important idea of a *symmetry*, or *covariance* transformation of the theory. These ideas were discussed briefly in Paper III [3].

In the example of classical physics, for all systems we have $\tau(S) = \mathbf{Sets}$ and $\Sigma_{\sigma,S}$ is a symplectic manifold, and the collection of all symplectic manifolds is a category. It would be elegant if we could assert that, in general, for a given theory-type the possible state objects in a given topos τ form the objects of an *internal* category in τ. However, to make such a statement would require a general theory of state objects and, at the moment, we do not have such a thing.

From a more conceptual viewpoint we note that the 'similarity' of our axioms to those of standard classical physics is reflected in the fact that (i) physical quantities are represented by arrows $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$; (ii) propositions are represented by sub-objects of $\Sigma_{\phi,S}$; and (iii) propositions are assigned truth values. Thus any theory satisfying these axioms 'looks' like classical physics, and has an associated neo-realist interpretation.

4 The General Scheme applied to Quantum Theory

4.1 Background Remarks

We now want to study the extent to which our 'rules' apply to the topos representation of quantum theory.

For a quantum system with (separable) Hilbert space $\mathcal{H}$, the appropriate topos (what we earlier called $\tau_\phi(S)$) is $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$: the category of presheaves over the category (actually, partially-ordered set) $\mathcal{V}(\mathcal{H})$ of unital, abelian von Neumann subalgebras of the algebra, $\mathcal{B}(\mathcal{H})$, of bounded operators on $\mathcal{H}$.

From a physical perspective we can think of the objects in $\mathcal{V}(\mathcal{H})$—i.e., the commutative subalgebras of $\mathcal{B}(\mathcal{H})$—as the 'contexts' (or 'world views', or 'windows on reality') with respect to which our generalised truth values of propositions are assigned. In the normal, instrumentalist interpretation of quantum theory, a context is therefore a collection of physical variables that can be measured simultaneously. The physical significance of this contextual logic is discussed at length in [7, 8, 9, 10, 11] and [2, 3].

[17]Care is needed not to confuse morphisms in the topos $\tau(S)$ with morphisms in the category $\mathcal{M}(\mathbf{Sys})$ of topoi. An arrow from the object $\tau(S)$ to itself in the category $\mathcal{M}(\mathbf{Sys})$ is a left-exact morphism in the topos $\tau(S)$. However, not every arrow in $\tau(S)$ need arise in this way, and an important role can be expected to be played by arrows of this type. A good example is when $\tau(S)$ is the category of sets, $\mathbf{Sets}$. Typically, $\tau_\phi(j) : \mathbf{Sets} \to \mathbf{Sets}$ is the identity, but there are many morphisms from an object O in $\mathbf{Sets}$ to itself: they are just the functions from O to O.

A particularly important object in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ is the *spectral presheaf* $\underline{\Sigma}$, where, for each V, $\underline{\Sigma}_V$ is defined to be the Gel'fand spectrum of the abelian algebra V. The sub-objects of $\underline{\Sigma}$ can be identified as the topos representations of propositions, just as the subsets of $\mathcal{S}$ represent propositions in classical physics.

In [3], several closely related choices for a quantity-value object $\mathcal{R}_\phi$ in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ were discussed. We concentrate here on the presheaf $\underline{\mathbb{R}^{\succeq}}$ of real-valued, order-reversing functions. Physical quantities $A : \Sigma \to \mathcal{R}$, which correspond to self-adjoint operators $\hat{A}$, are represented by natural transformations/arrows $\breve{\delta}^o(A) : \underline{\Sigma} \to \underline{\mathbb{R}^{\succeq}}$. The mapping $\hat{A} \mapsto \breve{\delta}^o(A)$ is injective. For brevity, we write $\breve{\delta}(A) := \breve{\delta}^o(A)$.[18] The arguments given in this section apply in similar form to the other possible choices for the quantity-value object.

Geometric Morphisms. Our constructions require a left-exact functor between two topoi, and one of the natural sources of such things is a *geometric morphism.* This fundamental concept in topos theory is defined as follows [6].

Definition 4.1 *A* geometric morphism $\phi : \mathcal{F} \to \mathcal{E}$ *between topoi* $\mathcal{F}$ *and* $\mathcal{E}$ *is a pair of functors* $\phi^* : \mathcal{E} \to \mathcal{F}$ *and* $\phi_* : \mathcal{F} \to \mathcal{E}$ *such that*

(i) $\phi^* \dashv \phi_*$*, i.e.,* ϕ^* *is left adjoint to* ϕ_**;*

(ii) ϕ^* *is left exact, i.e., it preserves all finite limits.*

Geometric morphisms are very important because they are the topos equivalent of continuous functions. More precisely, if X and Y are topological spaces, then any continuous function $f : X \to Y$ induces a geometric morphism between the topoi $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ of sheaves on X and Y respectively. In fact, just as the arrows in the category of topological spaces are continuous functions, so in any category whose objects are topoi, the arrows are normally defined to be geometric morphisms.

The key result for us is the following theorem ([6] p359):

Theorem 4.1 *If* $\varphi : \mathcal{C} \to \mathcal{D}$ *is a functor between categories* $\mathcal{C}$ *and* $\mathcal{D}$*, then it induces a geometric morphism (also denoted* φ*)*

$$\varphi : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \tag{4.1}$$

for which the functor $\varphi^* : \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ *takes a functor* $\underline{F} : \mathcal{D} \to \mathbf{Sets}$ *to the functor*

$$\varphi^*(\underline{F}) := \underline{F} \circ \varphi^{\mathrm{op}} \tag{4.2}$$

from $\mathcal{C}$ *to* $\mathbf{Sets}$*.*

In addition, φ^* *has a left adjoint* $\varphi_!$*; i.e.,* $\varphi_! \dashv \varphi^*$*.*

[18] Note that this is *not* the same as the convention used in paper III [3], where $\breve{\delta}(A)$ denoted a different natural transformation.

The morphism $\varphi^* : \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is called the *inverse image part* of the geometric morphism φ; the morphism $\varphi_* : \mathbf{Sets}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathcal{D}^{\mathrm{op}}}$ is called the *direct image part*.

We will use this important theorem in several crucial places.

4.2 The Translation Representation for a Disjoint Sum of Quantum Systems

Let **Sys** be a category whose objects are systems that can be treated using quantum theory. Let $\mathcal{L}(S)$ be the local language of a system S in **Sys** whose quantum Hilbert space is denoted $\mathcal{H}_S$. We assume that to each function symbol, $A : \Sigma \to \mathcal{R}$, in $\mathcal{L}(S)$ there is associated a self-adjoint operator $\hat{A} \in \mathcal{B}(\mathcal{H}_S)$,[19] and that the map

$$
\begin{aligned}
F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big) &\to \mathcal{B}(\mathcal{H})_{\mathrm{sa}} &\quad (4.3)\\
A &\mapsto \hat{A} &\quad (4.4)
\end{aligned}
$$

is injective (but not necessarily surjective, as we will see in the case of a disjoint sum of quantum systems).

We consider first arrows of the form

$$S_1 \xrightarrow{i_1} S_1 \sqcup S_2 \xleftarrow{i_2} S_2 \tag{4.5}$$

from the components S_1, S_2 to a disjoint sum $S_1 \sqcup S_2$; for convenience we write $i := i_1$. The systems S_1, S_2 and $S_1 \sqcup S_2$ have the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2$, respectively.

As always, the translation $\mathcal{L}(i)$ goes in the opposite direction to the arrow i, so

$$\mathcal{L}(i) : F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \to F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big). \tag{4.6}$$

Then our first step is find an 'operator translation' from the relevant self-adjoint operators in $\mathcal{H}_1 \oplus \mathcal{H}_2$ to those in $\mathcal{H}_1$,

To do this, let A be a function symbol in $F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big)$. In Section 2.2.1, we argued that $F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big) \simeq F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big) \times F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big)$ (as in (2.1)), and hence we introduce the notation $A = \langle A_1, A_2 \rangle$, where $A_1 \in F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ and $A_2 \in F_{\mathcal{L}(S_2)}\big(\Sigma, \mathcal{R}\big)$. It is then natural to assume that the quantisation scheme is such that the operator, $\hat{A}$, on $\mathcal{H}_1 \oplus \mathcal{H}_2$ can be decomposed as $\hat{A} = \hat{A}_1 \oplus \hat{A}_2$, where the operators $\hat{A}_1$ and $\hat{A}_2$ are defined on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and correspond to the function symbols A_1 and A_2.[20] Then the obvious operator translation is $\hat{A} \mapsto \hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{sa}$.

We now consider the general rules in the Definition 3.2 and see to what extent they apply in the example of quantum theory.

[19] More specifically, one could postulate that the elements of $F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$ are associated with self-adjoint operators in some unital von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_S)$.

[20] It should be noted that our scheme does not use all the self-adjoint operators on the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$: only the 'block diagonal' operators of the form $\hat{A} = \hat{A}_1 \oplus \hat{A}_2$ arise.

1. As we have stated several times, the topos $\tau_\phi(S)$ associated with a quantum system S is

$$\tau_\phi(S) = \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_S)^{op}}. \tag{4.7}$$

Thus (i) the objects of the category $\mathcal{M}(\mathbf{Sys})$ are topoi of the form $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_S)^{op}}$, $S \in \mathrm{Ob}(\mathbf{Sys})$; and (ii) the arrows between two topoi are defined to be left-exact functors. In particular, to each arrow $j : S_1 \to S$ in $\mathbf{Sys}$ there must correspond a left-exact functor $\tau_\phi(j) : \tau_\phi(S) \to \tau_\phi(S_1)$. Of course, the existence of these functors in the quantum case has yet to be shown.

2. The realisation $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$ of the language $\mathcal{L}(S)$ in the topos $\tau_\phi(S)$ is given as follows. First, we define the state object $\Sigma_{\phi,S}$ to be the spectral presheaf, $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_S)}$, over $\mathcal{V}(\mathcal{H}_S)$, the context category of $\mathcal{B}(\mathcal{H}_S)$. To keep the notation brief, we will denote[21] $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_S)}$ as $\underline{\Sigma}^{\mathcal{H}_S}$.

Furthermore, we define the quantity-value object, $\mathcal{R}_{\phi,S}$, to be the presheaf $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_S}$ that was defined in paper III [3]. Finally, we define

$$A_{\phi,S} := \breve{\delta}(A), \tag{4.8}$$

for all $A \in F_{\mathcal{L}(S)}\big(\Sigma, \mathcal{R}\big)$. Here $\breve{\delta}(A) : \underline{\Sigma}^{\mathcal{H}_S} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_S}$ is constructed using the Gel'fand transforms of the (outer) daseinisation of $\hat{A}$, for details see paper III.

3. The truth object $\mathbb{T}^\psi$ corresponding to a pure state ψ was discussed in paper II [2].

4. Let $\mathcal{H} = \mathbb{C}$ be the one-dimensional Hilbert space, corresponding to the trivial quantum system 1. There is exactly one abelian subalgebra of $\mathcal{B}(\mathbb{C}) \simeq \mathbb{C}$, namely $\mathbb{C}$ itself. This leads to

$$\tau_\phi(1_{\mathbf{Sys}}) = \mathbf{Sets}^{\{*\}} \simeq \mathbf{Sets} = 1_{\mathcal{M}(\mathbf{Sys})}. \tag{4.9}$$

5. Let $A \in F_{\mathcal{L}(S_1 \sqcup S_2)}\big(\Sigma, \mathcal{R}\big)$ be a function symbol for the system $S_1 \sqcup S_2$. Then, as discussed above, A is of the form $A = \langle A_1, A_2\rangle$ (compare equation (2.1)), which corresponds to a self-adjoint operator $\hat{A}_1 \oplus \hat{A}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)_{\mathrm{sa}}$. The topos representation of A is the natural transformation $\breve{\delta}(\langle A_1, A_2\rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$, which is defined at each stage $V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ as

$$\begin{aligned} \breve{\delta}(\langle A_1, A_2\rangle)_V : \underline{\Sigma}_V^{\mathcal{H}_1 \oplus \mathcal{H}_2} &\to \underline{\mathbb{R}^{\succeq}}_V^{\mathcal{H}_1 \oplus \mathcal{H}_2} \\ \lambda &\mapsto \{V' \mapsto \lambda|_{V'}(\delta(\hat{A}_1 \oplus \hat{A}_2)_{V'}) \mid V' \subseteq V\} \end{aligned} \tag{4.10}$$

where the right hand side (4.10) denotes an order-reversing function.

We will need the following:

Lemma 4.2 *Let $\hat{A}_1 \oplus \hat{A}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)_{\mathrm{sa}}$, and let $V = V_1 \oplus V_2 \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ such that $V_1 \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$ and $V_2 \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_2))$. Then*

$$\delta(\hat{A}_1 \oplus \hat{A}_2)_V = \delta(\hat{A}_1)_{V_1} \oplus \delta(\hat{A}_2)_{V_2}. \tag{4.11}$$

[21] Presheaves are always denoted by symbols that are underlined.

Proof. Every projection $\hat{Q} \in V$ is of the form $\hat{Q} = \hat{Q}_1 \oplus \hat{Q}_2$ for unique projections $\hat{Q}_1 \in \mathcal{P}(\mathcal{H}_1)$ and $\hat{Q}_2 \in \mathcal{P}(\mathcal{H}_2)$. Let $\hat{P} \in \mathcal{P}(\mathcal{H})$ be of the form $\hat{P} = \hat{P}_1 \oplus \hat{P}_2$ such that $\hat{P}_1 \in \mathcal{P}(\mathcal{H}_1)$ and $\hat{P}_2 \in \mathcal{P}(\mathcal{H}_1)$. The largest projection in V smaller than or equal to $\hat{P}$, i.e., the inner daseinisation of $\hat{P}$ to V, is

$$\delta^i(\hat{P})_V = \hat{Q}_1 \oplus \hat{Q}_2, \tag{4.12}$$

where $\hat{Q}_1 \in \mathcal{P}(V_1)$ is the largest projection in V_1 smaller than or equal to $\hat{P}_1$, and $\hat{Q}_2 \in \mathcal{P}(V_2)$ is the largest projection in V_2 smaller than or equal to $\hat{P}_2$, so

$$\delta^i(\hat{P})_V = \delta(\hat{P}_1)_{V_1} \oplus \delta(\hat{P}_2)_{V_2}. \tag{4.13}$$

This implies $\delta(\hat{A} \oplus \hat{B})_V = \delta(\hat{A})_{V_1} \oplus \delta(\hat{B})_{V_2}$, since (outer) daseinisation of a self-adjoint operator just means inner daseinisation of the projections in its spectral family, and all the projections in the spectral family of $\hat{A} \oplus \hat{B}$ are of the form $\hat{P} = \hat{P}_1 \oplus \hat{P}_2$. $\blacksquare$

As discussed in Section 3, in order to mimic the construction that we have in the classical case, we need to pull back the arrow/natural transformation $\breve{\delta}(\langle A_1, A_2 \rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$ to obtain an arrow from $\underline{\Sigma}^{\mathcal{H}_1}$ to $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$. Since we decided that the translation on the level of operators sends $\hat{A}_1 \oplus \hat{A}_2$ to $\hat{A}_1$, we expect that this arrow from $\underline{\Sigma}^{\mathcal{H}_1}$ to $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$ is $\breve{\delta}(A_1)$. We will now show how this works.

The presheaves $\underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$ and $\underline{\Sigma}^{\mathcal{H}_1}$ lie in different topoi, and in order to 'transform' between them we need we need a (left-exact) functor from the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{op}}$ to the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\mathrm{op}}}$: this is the functor $\tau_\phi(j) : \tau_\phi(S) \to \tau_\phi(S_1)$ in (3.17). One natural place to look for such a functor is as the inverse-image part of a geometric morphism from $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{op}}$ to $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\mathrm{op}}}$. According to Theorem 4.1, one source of such a geometric morphism, μ, is a functor

$$m : \mathcal{V}(\mathcal{H}_1) \to \mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2), \tag{4.14}$$

and the obvious choice for this is

$$m(V) := V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2} \tag{4.15}$$

for all $V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$. This function from $\mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$ to $\mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ is clearly order preserving, and hence m is a genuine functor.

Let μ denote the geometric morphism induced by m. The inverse-image functor of μ is given by

$$\begin{aligned} \mu^* : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\mathrm{op}}} &\to \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\mathrm{op}}} &(4.16)\\ \underline{F} &\mapsto \underline{F} \circ m^{\mathrm{op}}. &(4.17) \end{aligned}$$

This means that, for all $V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$, we have

$$(\mu^* \underline{F}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{F}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{m(V)} = \underline{F}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}. \tag{4.18}$$

For example, for the spectral presheaf we get

$$(\mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{m(V)} = \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}. \tag{4.19}$$

This is the functor that is denoted $\tau_\phi(j) : \tau_\phi(S_1) \to \tau_\phi(S)$ in (3.17).

We next need to find an arrow $\phi(i) : \underline{\Sigma}^{\mathcal{H}_1} \to \mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$ that is the analogue of the arrow $\phi(j) : \Sigma_{\phi, S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ in (3.17).

For each V, the set $(\mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}$ contains two types of spectral elements λ: the first type are those λ such that $\lambda(\hat{0}_{\mathcal{H}_1} \oplus \hat{1}_{\mathcal{H}_2}) = 0$. Then, clearly, there is some $\tilde{\lambda} \in \underline{\Sigma}^{\mathcal{H}_1}_V$ such that $\tilde{\lambda}(\hat{A}) = \lambda(\hat{A} \oplus \hat{0}_{\mathcal{H}_2}) = \lambda(\hat{A} \oplus \hat{1}_{\mathcal{H}_2})$ for all $\hat{A} \in V_{\rm sa}$. The second type of spectral elements $\lambda \in \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}$ are such that $\lambda(\hat{0}_{\mathcal{H}_1} \oplus \hat{1}_{\mathcal{H}_2}) = 1$. In fact, there is exactly one such λ, and we denote it by λ_0. This shows that $\underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}} \simeq \underline{\Sigma}^{\mathcal{H}_1}_V \cup \{\lambda_0\}$. Accordingly, at each stage V, the mapping $\phi(i)$ sends each $\tilde{\lambda} \in \underline{\Sigma}^{\mathcal{H}_1}_V$ to the corresponding $\lambda \in \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}$.

The presheaf $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$ is given at each stage $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ as the order-reversing functions $\nu : \downarrow\! W \to \mathbb{R}$, where $\downarrow\! W$ denotes the set of unital, abelian von Neumann subalgebras of W. Let $W = V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}$. Clearly, there is a bijection between the sets $\downarrow\! W \subset \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ and $\downarrow\! V \subset \mathrm{Ob}(\mathcal{V}(\mathcal{H}))$. We can thus identify

$$(\mu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}} \simeq \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}_V \tag{4.20}$$

for all $V \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}))$. This gives an isomorphism $\beta_\phi(i) : \mu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$, which corresponds to the arrow $\beta_\phi(j) : \mathcal{R}_{\phi, S_1} \to \tau_\phi(j)(\mathcal{R}_{\phi,S})$ in (3.17).

Now consider the arrow $\breve{\delta}(\langle A_1, A_2 \rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$. This is the analogue of the arrow $A_{\phi,S} : \Sigma_{\phi,S} \to \mathcal{R}_{\phi,S}$ in (3.17). At each stage $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$, this arrow is given by the (outer) daseinisation $\delta(\hat{A}_1 \oplus \hat{A}_2)_{W'}$ for all $W' \in \downarrow\! W$. According to Lemma 4.2, we have

$$\delta(\hat{A}_1 \oplus \hat{A}_2)_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_V \oplus \delta(A_2)_{\mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_V \oplus \max(\mathrm{sp}(\hat{\mathrm{A}}_2))\hat{1}_{\mathcal{H}_2} \tag{4.21}$$

for all $V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2} \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$. This makes clear how the arrow

$$\mu^*(\breve{\delta}(\langle A_1, A_2 \rangle)) : \mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \to \mu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \tag{4.22}$$

is defined. Our conjectured pull-back/translation representation is

$$\phi(\mathcal{L}(i))\big(\breve{\delta}(\langle A_1, A_2 \rangle)\big) := \phi(i) \circ \mu^*(\breve{\delta}(\langle A_1, A_2 \rangle)) \circ \beta_\phi(i) : \underline{\Sigma}^{\mathcal{H}_1} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}. \tag{4.23}$$

Using the definitions of $\phi(i)$ and $\beta_\phi(i)$, it becomes clear that

$$\phi(i) \circ \mu^*(\breve{\delta}(\langle A_1, A_2 \rangle)) \circ \beta_\phi(i) = \breve{\delta}(A_1). \tag{4.24}$$

Hence, the commutativity condition in (3.18) is satisfied for arrows in **Sys** of the form $i_{1,2} : S_{1,2} \to S_1 \sqcup S_2$.

4.3 The Translation Representation for Composite Quantum Systems

We now consider arrows in **Sys** of the form

$$S_1 \overset{p_1}{\leftarrow} S_1 \diamond S_2 \overset{p_2}{\rightarrow} S_1, \tag{4.25}$$

where the quantum systems S_1, S_2 and $S_1 \diamond S_2$ have the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$, respectively.[22]

The canonical translation[23] $\mathcal{L}(p_1)$ between the languages $\mathcal{L}(S_1)$ and $\mathcal{L}(S_1 \diamond S_2)$ (see Section 2.2.2) is such that if A_1 is a function symbol in $F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$, then the corresponding operator $\hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{\rm sa}$ will be 'translated' to the operator $\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. By assumption, this corresponds to the function symbol $A_1 \diamond 1$ in $F_{\mathcal{L}(S_1 \diamond S_2)}\big(\Sigma, \mathcal{R}\big)$.

Operator entanglement and translations. We should be cautious about what to expect from this translation when we represent a physical quantity $A : \Sigma \to \mathcal{R}$ in $F_{\mathcal{L}(S_1)}\big(\Sigma, \mathcal{R}\big)$ by an arrow between presheaves, since there are no canonical projections

$$\mathcal{H}_1 \leftarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_2, \tag{4.26}$$

and hence no canonical projections

$$\underline{\Sigma}^{\mathcal{H}_1} \leftarrow \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \rightarrow \underline{\Sigma}^{\mathcal{H}_2} \tag{4.27}$$

from the spectral presheaf of the composite system to the spectral presheaves of the components.[24]

This is the point where a form of *entanglement* enters the picture. The spectral presheaf $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is a presheaf over the context category $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Clearly, the context category $\mathcal{V}(\mathcal{H}_1)$ can be embedded into $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the mapping $V_1 \mapsto V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$, and likewise $\mathcal{V}(\mathcal{H}_2)$ can be embedded into $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. But not every $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ is of the form $V_1 \otimes V_2$.

[22] As usual, the composite system $S_1 \diamond S_2$ has as its Hilbert space the tensor product of the Hilbert spaces of the components.

[23] As discussed in Section 2.2.2, this translation, $\mathcal{L}(p_1)$, transforms a physical quantity A_1 of system S_1 into a physical quantity $A_1 \diamond 1$, which is the 'same' physical quantity but now seen as a part of the composite system $S_1 \diamond S_2$. The symbol 1 is the trivial physical quantity: it is represented by the operator $\hat{1}_{\mathcal{H}_2}$.

[24] On the other hand, in the classical case, there *are* canonical projections

$$\Sigma_{\sigma,S_1} \leftarrow \Sigma_{\sigma,S_1 \diamond S_2} \rightarrow \Sigma_{\sigma,S_2} \tag{4.28}$$

because the symplectic manifold $\Sigma_{\sigma,S_1 \diamond S_2}$ that represents the composite system is the cartesian product $\Sigma_{\sigma,S_1 \diamond S_2} = \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$, which is a product in the categorial sense and hence comes with canonical projections.

This comes from the fact that not all vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$ are of the form $\psi_1 \otimes \psi_2$, hence not all projections in $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are of the form $\hat{P}_{\psi_1} \otimes \hat{P}_{\psi_2}$, which in turn implies that not all $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ are of the form $V_1 \otimes V_2$. There are more contexts, or world-views, available in $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ than those coming from $\mathcal{V}(\mathcal{H}_1)$ and $\mathcal{V}(\mathcal{H}_2)$. We call this 'operator entanglement'.

The topos representative of $\hat{A}_1$ is $\breve{\delta}(A_1) : \underline{\Sigma}^{\mathcal{H}_1} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$, and the representative of $\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2}$ is $\breve{\delta}(A_1 \diamond 1) : \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$. At subalgebras $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ which are *not* of the form $W = V_1 \otimes V_2$ for any $V_1 \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$ and $V_2 \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_2))$, the daseinised operator $\delta(\hat{A}_{1W} \otimes \hat{1}_{\mathcal{H}_2}) \in W_{\mathrm{sa}}$ will not be of the form $\delta(\hat{A}_1)_{V_1} \otimes \delta(\hat{A}_1)_{V_2}$.[25] On the other hand, it is easy to see that $\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_W = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$ if $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$.

Given a physical quantity A_1, represented by the arrow $\breve{\delta}(A_1) : \underline{\Sigma}^{\mathcal{H}_1} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$, we can (at best) expect that the translation of this arrow into an arrow from $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ to $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ coincides with the arrow $\breve{\delta}(A_1 \diamond 1)$ on the '*image*' of $\underline{\Sigma}^{\mathcal{H}_1}$ in $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$. This image will be constructed below using a certain geometric morphism. As one might expect, the image of $\underline{\Sigma}^{\mathcal{H}_1}$ is a presheaf $\underline{P}$ on $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\underline{P}_{V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}} \simeq \underline{\Sigma}^{\mathcal{H}_1}_{V_1}$ for all $V_1 \in \mathcal{V}(\mathcal{H}_1)$, i.e., the presheaf $\underline{P}$ can be identified with $\underline{\Sigma}^{\mathcal{H}_1}$ exactly on the image of $\mathcal{V}(\mathcal{H}_1)$ in $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ under the embedding $V_1 \mapsto V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$. At these stages, the translation of $\breve{\delta}(A_1)$ will coincide with $\breve{\delta}(A_1 \diamond 1)$. At other stages $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the translation cannot be expected to be the same natural transformation as $\breve{\delta}(A \diamond 1)$ in general.

A geometrical morphism and a possible translation. The most natural approach to a translation is the following. Let $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$, and define $V_W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1))$ to be the *largest* subalgebra of $\mathcal{B}(\mathcal{H}_1)$ such that $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ is a subalgebra of W. Depending on W, V_W may, or may not, be the trivial subalgebra $\mathbb{C}\hat{1}_{\mathcal{H}_1}$. We note that if $W' \subseteq W$, then

$$V_{W'} \subseteq V_W, \tag{4.29}$$

but $W' \subset W$ only implies $V_{W'} \subseteq V_W$.

The trivial algebra $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ is not an object in the category $\mathcal{V}(\mathcal{H}_1)$. This is why we introduce the 'augmented context category' $\mathcal{V}(\mathcal{H}_1)_*$, whose objects are those of $\mathcal{V}(\mathcal{H}_1)$ united with $\mathbb{C}\hat{1}_{\mathcal{H}_1}$, and with the obvious morphisms ($\mathbb{C}\hat{1}_{\mathcal{H}_1}$ is a subalgebra of all $V \in \mathcal{V}(\mathcal{H}_1)$).

Then there is a functor $n : \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{V}(\mathcal{H}_1)_*$, defined as follows. On objects,

$$\begin{aligned} n : \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)) &\to \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1)_*) \\ W &\mapsto V_W, \end{aligned} \tag{4.30}$$

and if $i_{W'W} : W' \to W$ is an arrow in $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we define $n(i_{W'W}) := i_{V_{W'}V_W}$ (an arrow in $\mathcal{V}(\mathcal{H}_1)_*$); if $V_{W'} = V_W$, then $i_{V_{W'}V_W}$ is the identity arrow id_{V_W}.

[25]Currently, it is even an open question if $\delta(\hat{A}_{1W} \otimes \hat{1}_{\mathcal{H}_2}) = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$ if $W = V_1 \otimes V_2$ for a non-trivial algebra V_2.

Now let

$$\nu : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\rm op}} \rightarrow \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\rm op}} \tag{4.31}$$

denote the geometric morphism induced by π. Then the (left-exact) inverse-image functor

$$\nu^* : \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\rm op}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\rm op}} \tag{4.32}$$

acts on a presheaf $\underline{F} \in \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\rm op}}$ in the following way. For all $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$, we have

$$(\nu^* \underline{F})_W = \underline{F}_{n^{\rm op}(W)} = \underline{F}_{V_W} \tag{4.33}$$

and

$$(\nu^* \underline{F})(i_{W'W}) = \underline{F}(i_{V_{W'}V_W}) \tag{4.34}$$

for all arrows $i_{W'W}$ in the category $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.[26]

In particular, for all $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, we have

$$\begin{aligned} (\nu^* \underline{\Sigma}^{\mathcal{H}_1})_W &= \underline{\Sigma}^{\mathcal{H}_1}_{V_W}, & (4.36)\\ (\nu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1})_W &= \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}_{V_W}. & (4.37) \end{aligned}$$

Since V_W can be $\mathbb{C}\hat{1}_{\mathcal{H}_1}$, we have to extend the definition of the spectral presheaf $\underline{\Sigma}^{\mathcal{H}_1}$ and the quantity-value presheaf $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$ such that they become presheaves over $\mathcal{V}(\mathcal{H}_1)_*$ (and not just $\mathcal{V}(\mathcal{H}_1)$). This can be done in a straightforward way: the Gel'fand spectrum $\underline{\Sigma}_{\mathbb{C}\hat{1}_{\mathcal{H}_1}}$ of $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ consists of the single spectral element λ_1 such that $\lambda_1(\hat{1}_{\mathcal{H}_1}) = 1$. Moreover, $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ has no subalgebras, so the order-reversing functions on this algebra correspond bijectively to the real numbers $\mathbb{R}$.

Using these equations, we see that the arrow $\breve{\delta}(A_1) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$ that corresponds to the self-adjoint operator $\hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{\rm sa}$ gives rise to the arrow

$$\nu^*(\breve{\delta}(A_1)) : \nu^* \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \nu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}. \tag{4.38}$$

In terms of our earlier notation, the functor $\tau_\phi(p_1) : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\rm op}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\rm op}}$ is ν^*, and the arrow in (4.38) is the arrow $\tau_\phi(j)(A_{\phi,S}) : \tau_\phi(j)(\Sigma_{\phi,S}) \rightarrow \tau_\phi(j)(\mathcal{R}_{\phi,S})$ in (3.17) with $j : S_1 \rightarrow S$ being replaced by $p : S_1 \diamond S_2 \rightarrow S_1$, which is the arrow in **Sys** whose translation representation we are trying to construct.

The next arrow we need is the one denoted $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi,S}) \rightarrow \mathcal{R}_{\phi,S_1}$ in (3.17). In the present case, we define $\beta_\phi(p) : \nu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ as follows. Let $\alpha \in (\nu^* \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1})_W \simeq \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}_{V_W}$ be an order-reversing real-valued function on $\downarrow V_W$. Then we

[26] We remark, although will not prove here, that the inverse-image presheaf $\nu^* \underline{F}$ coincides with the direct image presheaf $\phi_* \underline{F}$ of $\underline{F}$ constructed from the geometric morphism ϕ induced by the functor

$$\begin{aligned} \kappa : \mathcal{V}(\mathcal{H}_1) &\rightarrow \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ V &\mapsto V \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}. \end{aligned} \tag{4.35}$$

Of course, the inverse image presheaf $\beta^* \underline{F}$ is much easier to construct.

define an order-reversing function $\beta_\phi(p)(\alpha) \in \underline{\mathbb{R}^{\succeq}}_W^{\mathcal{H}_1\otimes\mathcal{H}_2}$ as follows. For all $W' \subseteq W$, let

$$[\beta_\phi(p)(\alpha)](W') := \alpha(V_{W'}) \tag{4.39}$$

which, by virtue of (4.29), is an order-reversing function and hence a member of $\underline{\mathbb{R}^{\succeq}}_W^{\mathcal{H}_1\otimes\mathcal{H}_2}$.

We also need an arrow in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1\otimes\mathcal{H}_2)^{\mathrm{op}}}$ from $\underline{\Sigma}^{\mathcal{H}_1\otimes\mathcal{H}_2}$ to $\nu^*\underline{\Sigma}^{\mathcal{H}_1}$, where $\nu^*\underline{\Sigma}^{\mathcal{H}_1}$ is defined in (4.36). This is the arrow denoted $\phi(j) : \Sigma_{\phi,S_1} \to \tau_\phi(j)(\Sigma_{\phi,S})$ in (3.17).

The obvious choice is to restrict $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1\otimes\mathcal{H}_2}$ to the subalgebra $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2} \subseteq W$, and to identify $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_1} \simeq V_W \otimes \hat{1}_{\mathcal{H}_1} \simeq V_W$ as von Neumann algebras, which gives $\underline{\Sigma}_{V_W\otimes\mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1\otimes\mathcal{H}_2} \simeq \underline{\Sigma}_{V_W}^{\mathcal{H}_1}$. Let

$$\begin{aligned} \phi(p)_W : \underline{\Sigma}_W^{\mathcal{H}_1\otimes\mathcal{H}_2} &\to \underline{\Sigma}_{V_W}^{\mathcal{H}_1} \\ \lambda &\mapsto \lambda|_{V_W} \end{aligned} \tag{4.40}$$

denote this arrow at stage W. Then

$$\beta_\phi(p) \circ \nu^*(\breve{\delta}(A_1)) \circ \phi(p) : \underline{\Sigma}^{\mathcal{H}_1\otimes\mathcal{H}_2} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1\otimes\mathcal{H}_2} \tag{4.41}$$

is a natural transformation which is defined for all $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ and all $\lambda \in W$ by

$$\Big(\beta_\phi(p) \circ \nu^*(\breve{\delta}(A_1)) \circ \phi(p)\Big)_W(\lambda) = \nu^*(\breve{\delta}(A))(\lambda|_{V_W}) \tag{4.42}$$

$$= \{V' \mapsto \lambda|_{V'}(\delta(\hat{A})_{V'}) \mid V' \subseteq V_W\}. \tag{4.43}$$

This is clearly an order-reversing real-valued function on the set $\downarrow\! W$ of subalgebras of W, i.e., it is an element of $\underline{\mathbb{R}^{\succeq}}_W^{\mathcal{H}_1\otimes\mathcal{H}_2}$. We define $\beta_\phi(p) \circ \nu^*(\breve{\delta}(A_1)) \circ \phi(p)$ to be the translation representation, $\phi(\mathcal{L}(p))(\breve{\delta}(A_1))$ of $\breve{\delta}(A_1)$ for the composite system.

Note that, by construction, for each W, the arrow $(\beta_\phi(p) \circ \nu^*(\breve{\delta}(A_1)) \circ \phi(p))_W$ corresponds to the self-adjoint operator $\delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \in W_{\mathrm{sa}}$, since

$$\lambda|_{V_W}(\delta(\hat{A}_1)_{V_W}) = \lambda(\delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2}) \tag{4.44}$$

for all $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1\otimes\mathcal{H}_2}$.

Comments on these results. This is about as far as we can get with the arrows associated with the composite of two quantum systems. The results above can be summarised in the equation

$$\phi(\mathcal{L}(p))(\breve{\delta}(A_1))_W = \breve{\delta}(A_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \tag{4.45}$$

for all contexts $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$. If $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ is of the form $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$, i.e., if W is in the image of the embedding of $\mathcal{V}(\mathcal{H}_1)$ into $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, then $V_W = V_1$ and the translation formula gives just what one expects: the arrow

$\breve{\delta}(A_1)$ is translated into the arrow $\breve{\delta}(A_1 \diamond 1)$ at these stages, since $\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_{V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$.[27]

If $W \in \mathrm{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ is not of the form $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$, then it is relatively easy to show that

$$\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_W \neq \delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \tag{4.46}$$

in general. Hence

$$\phi(\mathcal{L}(p))(\breve{\delta}(A_1)) \neq \breve{\delta}(A_1 \diamond 1), \tag{4.47}$$

whereas, intuitively, one might have expected equality. Thus the 'commutativity' condition (3.7) is not satisfied.

In fact, there appears to be *no* operator $\hat{B} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $\phi(\mathcal{L}(p))(\breve{\delta}(A_1)) = \breve{\delta}(B)$. Thus the quantity, $\beta_\phi(p) \circ \nu^*(\breve{\delta}(A_1)) \circ \phi(p)$, that is our conjectured pull-back, is an arrow in $\mathrm{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\mathrm{op}}}}\big(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}, \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \otimes \mathcal{H}_2}\big)$ that is not of the form $A_{\phi, S_1 \diamond S_2}$ for any physical quantity $A \in F_{\mathcal{L}(S_1 \diamond S_2)}\big(\Sigma, \mathcal{R}\big)$.

Our current understanding is that this translation is 'as good as possible': the arrow $\breve{\delta}(A_1) : \underline{\Sigma}^{\mathcal{H}_1} \to \underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1}$ is translated into an arrow from $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ to $\underline{\mathbb{R}^{\succeq}}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ that coincides with $\breve{\delta}(A_1)$ on those part of $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ that can be identified with $\underline{\Sigma}^{\mathcal{H}_1}$. But $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is much larger, and it is not simply a product of $\underline{\Sigma}^{\mathcal{H}_1}$ and $\underline{\Sigma}^{\mathcal{H}_2}$. The context category $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ underlying $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is much richer than a simple product of $\mathcal{V}(\mathcal{H}_1)$ and $\mathcal{V}(\mathcal{H}_2)$. This is due to a kind of operator entanglement. A translation can at best give a faithful picture of an arrow, but it cannot possibly 'know' about the more complicated contextual structure of the larger category.

Clearly, both technical and interpretational work remain to be done.

5 Conclusions

In the previous three papers we have developed the idea that, for a given theory-type, the theory of a particular system, S, is to be constructed in the framework of a certain, system-dependent, topos. The central idea is that a local language, $\mathcal{L}(S)$, is attached to each system S, and that the application of a given theory-type to S is equivalent to finding a representation, ϕ, of $\mathcal{L}(S)$ in a topos $\tau_\phi(S)$.

Physical quantities are represented by arrows in the topos from the state object $\Sigma_{\phi,S}$ to the quantity-value object $\mathcal{R}_{\phi,S}$, and propositions are represented by sub-objects of the state object. The idea of a 'truth sub-object' of $P\Sigma_{\phi,S}$ then leads to a neo-realist interpretation of propositions in which each proposition is assigned a truth value that is a global element of the sub-object classifier $\Omega_{\tau_\phi(S)}$. In general, neo-realist statements about the world/system S are to be expressed in the internal language of

[27] To be precise, both the translation $\phi(\mathcal{L}(p))(\breve{\delta}(A_1))_W$, given by (4.45), and $\breve{\delta}(A \diamond 1)_W$ are mappings from $\underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ to $\underline{\mathbb{R}^{\succeq}}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$. Each $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is mapped to an order-reversing function on $\downarrow\!W$. The mappings $\phi(\mathcal{L}(p))(\breve{\delta}(A_1))_W$ and $\breve{\delta}(A \diamond 1)_W$ coincide at all $W' \in \downarrow\!W$ that are of the form $W' = V' \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$.

the topos $\tau_\phi(S)$. Underlying this is the intuitionistic, deductive logic provided by the local language $\mathcal{L}(S)$.

Every classical system uses the same topos, namely the topos of sets. However, in general, the topos will be system dependent, which leads to the problem of understanding how the topoi for a class of systems behave under the action of taking a sub-system, or combining a pair of systems to give a single composite system. In the present paper, we have presented a set of axioms that capture the general ideas we are trying to develop. Of course, these axioms are not cast in stone, and are still partly 'experimental' in nature. However, we have shown that classical physics exactly fits our suggested scheme, and that quantum physics 'almost' does: 'almost' because of the issues concerning the translation representation of the arrows associated with compositions of systems that were discussed in Section 4.3.

Is there 'un gros topos'? It is clear that there are many topics for future research, both in regard to the first three papers, and to the present one. As far as the present paper is concerned, a question that is of particular interest is if there is a *single* topos within which all systems of a given theory-type can be discussed. For example, in the case of quantum theory the relevant topoi are of the form $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$, where $\mathcal{H}$ is a Hilbert space, and the question is whether all such topoi can be gathered together to form a single topos (what Grothendieck termed 'un gros topos') within which all quantum systems can be discussed.

There are well-known examples of such constructions in the mathematical literature. For example, the category, $\mathrm{Sh}(X)$, of sheaves on a topological space X is a topos, and there are collections $\mathbf{T}$ of topological spaces which form a Grothendieck site, so that the topos $\mathrm{Sh}(\mathbf{T})$ can be constructed. A particular object in $\mathrm{Sh}(\mathbf{T})$ will then be a sheaf over $\mathbf{T}$ whose stalk over any object X in $\mathbf{T}$ will be the topos $\mathrm{Sh}(X)$.

For our purposes, the ideal situation would be if the various categories of systems, $\mathbf{Sys}$, can be chosen in such a way that $\mathcal{M}(\mathbf{Sys})$ is a site. Then the topos of sheaves, $\mathrm{Sh}(\mathcal{M}(\mathbf{Sys}))$, over this site would provide a common topos in which all systems of this theory type—i.e., the objects of $\mathbf{Sys}$—can be discussed. We do not know if this is possible, and it is a natural subject for future study.

Some more speculative lines of future research. At a conceptual level, one motivating desire for the entire research programme was to find a formalism that would always give some sort of 'realist' interpretation, even in the case of quantum theory which is normally presented in an instrumentalist way. But this particular example raises an interesting point because the neo-realist interpretation takes place in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$, whereas the instrumentalist interpretation works in the familiar topos $\mathbf{Sets}$ of sets, and one might wonder how universal is the use of a pair of topoi in this way.

Another, related, issue concerns the representation of the $\mathcal{PL}(S)$-propositions of the form "$A \,\varepsilon\, \Delta$" discussed in paper II. This serves as a bridge between the 'external' world of a background spatial structure, and the internal world of the topos. This link is not present with the $\mathcal{L}(S)$ language whose propositions are purely internal terms of type Ω of the form '$A(\tilde{s}) \in \tilde{\Delta}$', as discussed in paper I. In a topos representation, ϕ, of $\mathcal{L}(S)$, these become propositions of the form '$A \in \Xi$', where Ξ is a sub-object of $\mathcal{R}_\phi$.

In general, if we have an example of our axioms working neo-realistically in a topos τ, one might wonder if there is an 'instrumentalist' interpretation of the same theory in a different topos, τ_i, say? Of course, the word 'instrumentalism' is used metaphorically here, and any serious consideration of such a pair (τ, τ_i) would require a lot of very careful thought.

However, if a pair (τ, τ_i) does exist, the question then arises of whether there is a *categorial* way of linking the neo-realist and instrumentalist interpretations: for example, via a functor $I : \tau \to \tau_i$. If so, is this related to some analogue of the daseinisation operation that produced the representation of the $\mathcal{PL}(S)$-propositions, "$A \,\varepsilon\, \Delta$" in quantum theory? Care is needed in discussing such issues since informal set theory is used as a meta-language in constructing a topos, and one has to be careful not to confuse this with the existence, or otherwise, of an 'instrumentalist' interpretation of any given representation.

If such a functor, $I : \tau \to \tau_i$, did exist then one could speculate on the possibility of finding an 'interpolating chain' of functors

$$\tau \to \tau^1 \to \tau^2 \to \cdots \to \tau^n \to \tau_i \tag{5.1}$$

which could be interpreted conceptually as corresponding to an interpolation between the philosophical views of realism and instrumentalism!

Even more speculatively one might wonder if "one person's realism is another person's instrumentalism". More precisely, given a pair (τ, τ_i) in the sense above, could there be cases in which the topos τ carries a neo-realism interpretation of a theory with respect to an instrumentalist interpretation in τ_i, whilst being the carrier of an instrumentalist interpretation with respect to the neo-realism of a 'higher' topos; and so on? For example, is there some theory whose 'instrumentalist manifestation' takes place in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$?

On the other hand, one might want to say that 'instrumentalist' interpretations always take place in the world of classical set theory, so that τ_i should always be chosen to be **Sets**. In any event, it would be interesting to study the quantum case more closely to see if there are any categorial relations between the formulation in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ and the instrumentalism interpretation in **Sets**. It can be anticipated that the action of daseinisation will play an important role here. We hope to say more about this in a later paper.

Implications for quantum gravity. A serious claim stemming from our work is that a successful theory of quantum gravity should be constructed in some topos $\mathcal{U}$—the 'topos of the universe'—that is *not* the topos of sets. All entities of physical interest will be represented in this topos, including models for space-time (if there are any at a fundamental level in quantum gravity) and, if relevant, loops, membranes etc. as well as incorporating the anticipated generalisation of quantum theory.

Such a theory of quantum gravity will have a neo-realist interpretation in the topos $\mathcal{U}$, and hence would be particularly useful in the context of quantum cosmology. However, in practice, physicists divide the world up into smaller, more easily handled, chunks, and each of them would correspond to what earlier we have called a 'system' and, correspondingly, would have its own topos. Thus $\mathcal{U}$ would be something like the 'gros topos' of the theory, and would combine together the individual 'sub-systems' in a categorial way. Of course, it is most unlikely that there is any preferred way of dividing the universe up into bite-sized chunks, but this is not problematic as the ensuing relativism would be naturally incorporated into the idea of a Grothendieck site.

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